20-th Brazilian Mathematical Olympiad 1998

Third Round

First Day – October 24

1. Let 15 positive intgers be given, all greater than 1 and smaller than 1998 and pairwise coprime. Prove that at least one of the 15 numbers is a prime number.

2. Let D be the midpoint of side AB of triangle ABC and E a point on side BC such that $BE = 2EC$. If $\angle CDA = \angle BAE$, find angle $\angle BAC$.

3. Two players A and B play the following game. First they choose two positive integers $n > 1$ (the number of rounds) and $t > 0$ (the maximum increment). In the first round, A chooses a positive integer m_1, and after that B chooses a positive integer n_1 different from m_1. In the k-th round ($2 \le k \le n$), A chooses an integer m_k satisfying $m_{k-1} < m_k \le m_{k-1} + t$. Then B chooses an integer n_k satisfying $n_{k-1} < n_k \le n_{k-1} + t$. A then wins $\gcd(m_k, n_{k-1})$ points and B wins $\gcd(m_k, n_k)$ points. The player with highest punctation at the end of the n rounds wins the game. If they have same punctuation, A is considered the winner. Given initial values n and t, find which player has a winning strategy.

Second Day – October 31

4. Two players A and B play the following game. First A chooses two distinct integers a and b, different from zero, and then B constructs a quadratic equation with coefficients a, b and 1998 (in some order – e.g. he may construct $1998x^2 + ax + b$ or $ax^2 + 1998x + b$, etc.). If the resulting equation has two distinct rational roots, then A is the winner. Prove that A can always win.

5. Find all functions $f : \mathbb{N} \to \mathbb{N}$ satisfying, for all $x \in \mathbb{N}$,
$$f(2f(x)) = x + 1998.$$

6. Two mathematicians in Berlin came across the intersection of streets Barbarossa and Martin Luther. They want to go to the intersection of the streets Meininger and Martin Luther. Unfortunately, they don't know either to which side of Barbarossa is Meininger, neither the distance (in blocks) of Meininger to Barbarossa. So, they have to go back and forth on Martin Luther till they find Meininger. What is the least positive real constant k such that, if the distance of Barbarossa to Meininger is N blocks, they can surely reach Meininger walking at most kN blocks? (The mathematicians cannot go separately.)

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