18-th Brazilian Mathematical Olympiad 1996

Final Round

First Day

1. Show that the equation
$$x^2+y^2+z^2=3xyz$$
has infinitely many solutions in positive integers (x,y,z).

2. Is there a set A of $n \geq 3$ points in the plane such that:
   (i) no three points in A are collinear;
   (ii) for any three points in A, the center of the circle passing through these points is also in A?

3. For $n \in \mathbb{N}$, let $f(n)$ be the smallest number of ones that can be used to represent n using ones (but not 11,111 etc.) and any number of the symbols $+$, $\cdot$, $($, $)$. For example, $80=(1+1+1+1+1)\cdot(1+1+1+1)\cdot(1+1+1+1)$, and therefore $f(80) \leq 13$. Show that
$$3\log_3 n \leq f(n) < 5\log_3 n \qquad \text{for all } n>1.$$

Second Day

4. Let D be a point on the side BC of an acute-angled triangle ABC different from B and C, O_1, O_2 be the circumcenters of triangles ABD and ACD, and O be the circumcenter of $\triangle AO_1O_2$. Find the locus of point O when D runs through the side BC.

5. A set of marriages is *unstable* if two persons who are not married to each other prefer each other to their spouses. For instance, if Alessandra and Daniel are married and Julia and Robinson are married, but Daniel prefers Julia to Alessandra, and Julia prefers Daniel to Robinson, then the set of marriages Alessandra–Daniel and Julia–Robinson is unstable. The set of marriages is *stable* if it is not unstable.

   Consider now a group of n boys and n girls. Each boy makes his own list ordering the n girls according to his preference and each girl lists the n boys according to her preference. Show that it is always possible to marry the n boys and the n girls obtaining a stable marriage set.

6. Consider the polynomial
$$T(x)=x^3+14x^2-2x+1.$$
Show that there exists a natural number $n>1$ such that $T^{(n)}(x)-x$ is divisible by 101 for all integers x, where $T^{(n)}(x)=\underbrace{T(T(\cdots T}_{n}(x)\cdots))$.

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