16-th Brazilian Mathematical Olympiad 1994

Final Round

First Day

1. We assign to each edge of a cube one of the numbers $1, 2, \ldots, 12$, no two edges having the same number. Then we write at each vertex the sum of the numbers at the edges having this vertex as an endpoint.

   (a) Show that it is not possible that the eight numbers at the vertices be all equal.

   (b) Can all the numbers be equal, if one of them is replaced by 13?

2. Consider a convex polygon, and consider all the circles passing through three consecutive vertices of the polygon. Prove that one of these circles contains the entire polygon.

3. We are given n identical objects with distinct weights. We also have a pair of scales. At each step we may put two objects, one on each scale, and compare their weights. Find, as a function of n, the minimum number of measures necessary to determine the heaviest and the lightest object.

Second Day

4. Suppose that a and b are positive real numbers such that

$$a^3 = a + 1 \quad \text{and} \quad b^6 = b + 3a.$$

   Show that $a > b$.

5. A sequence of integers is called *super-integer* when the first term is a one-digit integer and every consequent term is obtained by adding a digit (which may be zero) to the left of the previous term of the sequence. For example, $(2, 32, 532, 7532, \ldots)$ is a super-integer sequence. The sequence $(0, 00, 000, 0000, \ldots)$ is called the zero sequence. The product of two super-integer sequences $(a_1, a_2, \ldots)$ and $(b_1, b_2, \ldots)$ is defined as $(S_1(a_1b_1), S_2(a_2b_2), \ldots)$, where $S_k(n)$ denotes the last k digits of n. Is it possible to have two nonzero super-integer sequences having as product the zero sequence?

6. In a triangle ABC, let R be the circumradius, r the inradius, and s the semiperimeter. Prove that $2R = s - r$ if and only if ABC is a right triangle.

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