26-th Brazilian Mathematical Olympiad 2004

Third Round

First Day

1. Let $ABCD$ be a convex quadrilateral. Prove that the incircles of triangles ABC, BCD, CDA and DAB have a common point if and only if $ABCD$ is a rhombus.

2. Determine all positive integers n for which it is possible to divide a triangle in n non-degenerate triangles in such a way that each of the vertices of the triangles is an endpoint of the same number of sides of the triangles.

3. Let $x_1, x_2, \dots, x_{2004}$ be a sequence of integers satisfying

$$x_{k+3} = x_{k+2} + x_{k+1}x_k \qquad \text{for each } k = 1, 2, \dots, 2001.$$

Can more than a half of the terms of this sequence be negative?

Second Day

4. Consider all 10×10 tables with entries $0, 1, \dots, 9$ which contain each of the numbers $0, \dots, 9$ exactly 10 times. Find the greatest n such that, in every such table, there is a row or column containing at least n distinct entries.

5. Consider the sequence (a_n) given by $a_0 = a_1 = a_2 = a_3 = 1$ and

$$a_n a_{n-4} = a_{n-1}a_{n-3} + a_{n-2}^2.$$

Prove that all its terms are integers.

6. Let a, b be real numbers. A function $f_{a,b} : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by

$$f_{a,b}(x, y) = (a - by - x^2, x).$$

For each $P \in \mathbb{R}^2$, define $f_{a,b}^0(P) = P$ and $f_{a,b}^{k+1}(P) = f_{a,b}(f_{a,b}^k(P))$, where k is a nonnegative integer. The set $per(a, b)$ of *periodic points* of $f_{a,b}$ is defined as the set of all points $P \in \mathbb{R}^2$ for which there exists $n \in \mathbb{N}$ such that $f_{a,b}^n(P) = P$.

For a fixed real b, show that the set $A_b = \{a \in \mathbb{R} \mid per(a, b) \neq \emptyset\}$ has the smallest element. Determine this smallest element.