25-th Brazilian Mathematical Olympiad 2003

Third Round

First Day

1. Determine the smallest prime number which divides $x^2+5x+23$ for some integer x.

2. Let S be a set of n elements. Determine the smallest positive integer k with the following property: Given any k distinct subsets $A_1, A_2, \dots, A_k$ of S, it is possible to choose signs $+$ and $-$ so that
$$S = A_1^{\pm} \cup A_2^{\pm} \cup \cdots \cup A_k^{\pm},$$
where $A_i^+ = A_i$ and $A_i^- = S \setminus A_i$ for each subset A_i.

3. Let $ABCD$ be a rhombus. Points E, F, G, H are given on sides AB, BC, CD, DA, respectively, so that the lines EF and GH are tangent to the incircle of the rhombus. Prove that the lines EH and FG are parallel.

Second Day

4. A circle k and a point A in its interior are given in the plane. Find points B, C, D on the circle such that the area of quadrilateral $ABCD$ is maximum possible.

5. Suppose that a function $f : \mathbb{R}^+ \to \mathbb{R}$ satisfies:

   (a) If $x < y$ then $f(x) < f(y)$;

   (b) $f\left(\frac{2xy}{x+y}\right) \geq \frac{f(x)+f(y)}{2}$ for all $x, y > 0$.

   Prove that there exists $x_0 > 0$ for which $f(x_0) < 0$.

6. A graph whose set of vertices V has n elements is called *excellent* if there are a set $D \in \mathbb{N}$ and an injective function $f : V \to \{1, 2, \dots, [n^2/4]\}$ such that two vertices p and q are joined by an edge if and only if $|f(p) - f(q)| \in D$. Show that there exists $n_0 \in \mathbb{N}$ such that for each $n \geq n_0$ there exist graphs with n vertices that are not excellent.

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