25-th Brazilian Mathematical Olympiad 2003

Third Round

First Day

1. Determine the smallest prime number which divides $x^2 + 5x + 23$ for some integer $x$.

2. Let $S$ be a set of $n$ elements. Determine the smallest positive integer $k$ with the following property: Given any $k$ distinct subsets $A_1, A_2, \ldots, A_k$ of $S$, it is possible to choose signs $+$ and $−$ so that

$$S = A_1^+ \cup A_2^+ \cup \cdots \cup A_k^+,$$

where $A_i^+ = A_i$ and $A_i^- = S \setminus A_i$ for each subset $A_i$.

3. Let $ABCD$ be a rhombus. Points $E, F, G, H$ are given on sides $AB, BC, CD, DA$, respectively, so that the lines $EF$ and $GH$ are tangent to the incircle of the rhombus. Prove that the lines $EH$ and $FG$ are parallel.

Second Day

4. A circle $k$ and a point $A$ in its interior are given in the plane. Find points $B, C, D$ on the circle such that the area of quadrilateral $ABCD$ is maximum possible.

5. Suppose that a function $f : \mathbb{R}^+ \to \mathbb{R}$ satisfies:

(a) If $x < y$ then $f(x) < f(y)$;

(b) $f \left( \frac{2xy}{x+y} \right) \geq \frac{f(x) + f(y)}{2}$ for all $x, y > 0$.

Prove that there exists $x_0 > 0$ for which $f(x_0) < 0$.

6. A graph whose set of vertices $V$ has $n$ elements is called excellent if there are a set $D \in \mathbb{N}$ and an injective function $f : V \to \{1, 2, \ldots, [n^2/4]\}$ such that two vertices $p$ and $q$ are joined by an edge if and only if $|f(p) - f(q)| \in D$. Show that there exists $n_0 \in \mathbb{N}$ such that for each $n \geq n_0$ there exist graphs with $n$ vertices that are not excellent.