The Belarusian Team Selection Tests 2000

First Test

1. Find the minimal number of cells on a 5×7 board that must be painted so that any cell which is not painted has exactly one neighboring (having a common side) painted cell.

2. Let P be a point inside a triangle ABC with $\angle C = 90^\circ$ such that $AP = AC$, and let M be the midpoint of AB and CH be the altitude. Prove that PM bisects $\angle BPH$ if and only if $\angle A = 60^\circ$.

3. Does there exist a function $f : \mathbb{N} \to \mathbb{N}$ such that
$$f(f(n-1)) = f(n+1) - f(n) \quad \text{for all } n \geq 2?$$

4. A closed pentagonal line is inscribed in a sphere of the diameter 1, and has all edges of length l. Prove that $l \leq \sin \dfrac{2\pi}{5}$.

Second Test

1. All vertices of a convex polyhedron are endpoints of exactly four edges. Find the minimal possible number of triangular faces of the polyhedron.

2. Real numbers a, b, c satisfy the equation
$$2a^3 - b^3 + 2c^3 - 6a^2b + 3ab^2 - 3ac^2 - 3bc^2 + 6abc = 0.$$
If $a < b$, find which of the numbers b, c is larger.

3. In the Cartesian plane, two integer points (a_1, b_1) and (a_2, b_2) are *connected* if (a_2, b_2) is one of the points $(-a_1, b_1 \pm 1)$, $(a_1 \pm 1, -b_1)$. Show that there exists an infinite sequence of integer points in which every integer point occurs, and every two consecutive points are connected.

4. In a triangle ABC with $AC = b \neq BC = a$, points E, F are taken on the sides AC, BC respectively such that $AE = BF = \dfrac{ab}{a+b}$. Let M and N be the midpoints of AB and EF respectively, and P be the intersection point of the segment EF with the bisector of $\angle ACB$. Find the ratio of the area of $CPMN$ to that of ABC.

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Third Test

1. In a triangle ABC, let $a = BC$, $b = AC$ and let m_a, m_b be the corresponding medians. Find all real numbers k for which the equality $m_a + ka = m_b + kb$ implies that $a = b$.

2. (a) Prove that $\{n\sqrt{3}\} > \dfrac{1}{n\sqrt{3}}$ for any positive integer n.

   (b) Is there a constant $c > 1$ such that $\{n\sqrt{3}\} > \dfrac{c}{n\sqrt{3}}$ for all $n \in \mathbb{N}$?

3. Each edge of a graph with 15 vertices is colored either red or blue in such a way that no three vertices are pairwise connected with edges of the same color. Determine the largest possible number of edges in the graph.

Fourth Test

1. Find all functions $f, g, h : \mathbb{R} \to \mathbb{R}$ such that
$$f(x + y^3) + g(x^3 + y) = h(xy) \quad \text{for all } x, y \in \mathbb{R}.$$

2. If M is a point inside a triangle ABC, prove that
$$\min\{MA, MB, MC\} + MA + MB + MC < AB + AC + BC.$$

3. Prove that for every positive integer N there exists an infinite arithmetic progression (a_k) such that:

   (i) each term is a positive integer and the common difference d is not divisible by 10;

   (ii) the sum of the decimal digits of each term is greater than N.

Fifth Test

1. Let AM and AL be the median and bisector of a triangle ABC $(M, L \in BC)$. If $BC = a$, $AM = m_a$, $AL = l_a$, prove the inequalities:

   (a) $a \tan\frac{\alpha}{2} \le 2m_a \le a \cot\frac{\alpha}{2}$ if $\alpha < \frac{\pi}{2}$, and
   $a \tan\frac{\alpha}{2} \ge 2m_a \ge a \cot\frac{\alpha}{2}$ if $\alpha > \frac{\pi}{2}$

   (b) $2l_a \le a \cot\frac{\alpha}{2}$.

2. Let n, k be positive integers such that n is not divisible by 3 and $k \ge n$. Prove that there exists a positive integer m that is divisible by n and the sum of whose digits in decimal representation is k.

3. Suppose that every integer has been given one of the colors red, blue, green, yellow. Let x and y be odd integers such that $|x| \neq |y|$. Show that there are two integers of the same color whose difference has one of the following values: x, y, $x+y$, $x-y$.

Sixth Test

1. Find the smallest natural number n for which it is possible to partition the set $M = \{1, 2, \dots, 40\}$ into n subsets $M_1, \dots, M_n$ so that none of the M_i contains elements a, b, c (not necessarily distinct) with $a + b = c$.

2. A positive integer $\overline{A_k \dots A_1 A_0}$ is called *monotonic* if $A_k \leq \cdots \leq A_1 \leq A_0$. Show that for any $n \in \mathbb{N}$ there is a monotonic perfect square with n digits.

3. Starting with an arbitrary pair (a, b) of vectors on the plane, we are allowed to perform the operations of the following two types:

   (1) To replace (a, b) with $(a + 2kb, b)$ for an arbitrary integer $k \neq 0$;

   (2) To replace (a, b) with $(a, b + 2ka)$ for an arbitrary integer $k \neq 0$.

   However, we must change the type of operetion in any step.

   (a) Is it possible to obtain $((1,0),(2,1))$ from $((1,0),(0,1))$, if the first operation is of the type (1)?

   (b) Find all pairs of vectors that can be obtained from $((1,0),(0,1))$ (the type of first operation can be selected arbitrarily).

Seventh Test

1. For any positive numbers a, b, c, x, y, z, prove the inequality

$$\frac{a^3}{x} + \frac{b^3}{y} + \frac{c^3}{z} \geq \frac{(a+b+c)^3}{3(x+y+z)}.$$

2. Let X be a variable point on the arc AB not containing C of the circumcircle k of a triangle ABC, and let O_1, O_2 be the incenters of the triangles CAX and CBX. Prove that the circumcircle of the triangle XO_1O_2 intersects k in a fixed point.

3. A game is played by $n \geq 2$ girls, everybody having a ball. Each of the $\binom{n}{2}$ pairs of players, in an arbitrary order, exchange the balls they have at that moment. The game is called *nice* if at the end nobody has her own ball, and it is called *tiresome* if at the end everybody has her initial ball. Determine the values of n for which there exists a nice game and those for which there exists a tiresome game.

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Eighth Test

1. The diagonals of a convex quadrilateral $ABCD$ with $AB = AC = BD$ intersect at P, and O and I are the circumcenter and incenter of $\triangle ABP$, respectively. Prove that if $O \neq I$ then OI and CD are perpendicular.

2. Prove that there exist two strictly increasing sequences (a_n) and (b_n) such that $a_n(a_n+1)$ divides b_n^2+1 for every natural number n.

3. Prove that the set of positive integers cannot be partitioned into three nonempty subsets such that for any two integers x, y taken from two different subsets, the number $x^2 - xy + y^2$ belongs to the third subset.

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