Belarussian Mathematical Olympiad 1995

Final Round

Category D

1. Mark six points in a plane so that any three of them are vertices of a non-degenerate isosceles triangle.

2. Find all positive integers n so that both n and $n+100$ have odd numbers of divisors.

3. Some students of a group were friends of some others. One day all students of the group take part in a picnic. During the picnic some friends had a quarrel with each other, but some other students became friends. After the picnic, the number of friends for each student changed by 1. Prove that the number of students in the group was even.

4. Given a triangle ABC, let K be the midpoint of AB and L a be point on AC such hat $AL = LC + CB$. Prove that $\angle KLB = 90^\circ$ if and only if $AC = 3CB$.

5. Two circles touch in M, and lie inside a rectangle $ABCD$. One of them touches the sides AB and AD, and the other one touches AD, BC, CD. The radius of the second circle is four times that of the first circle. Find the ratio in which the common tangent of the circles in M divides AB and CD.

6. Let p and q be distinct positive integers. Prove that at least one of the equations $x^2 + px + q = 0$ and $x^2 + qx + p = 0$ has a real root.

7. The expression $1*2*3*4*5*6*7*8*9$ is written on a blackboard. Bill and Peter play the following game. They replace $*$ by $+$ or $\cdot$, making their moves in turn, and one of them can use only $+$, while the other one can use only $\cdot$. At the beginning Bill selects the sign he will use, and he tries to make the result an even number. Peter tries to make the result an odd number. Prove that Peter can always win.

8. Five numbers $1,2,3,4,5$ are written on a blackboard. One may erase any two numbers a and b and write ab and $a+b$ instead. Afterwards, he/she can continue doing the same operation. Can the numbers $21,27,64,180,540$ be obtained after a finite number of steps?

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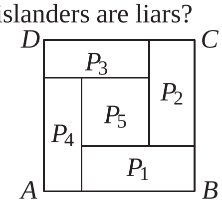
Category C

1. Six distinct numbers are given. Bill calculates the sum of each two of these numbers. What is the largest possible number of distinct primes he can obtain?

2. In a star-shaped closed broken line ABCDEA, AB meets CD and DE at P and Q, BC meets DE and EA at R and S, and CD meets EA at T, respectively, and $AP = QB$, $BR = SC$, $CT = PD$, $DQ = RE$. Prove that $ES = TA$.

3. Every inhabitant of a certain island either always tells the truth or always lies. A new governor wants to determine who is a liar on this island. For this purpose, every day he gathers a group of islanders and asks each of them, how many liars are there in the group. What the minimum number of days would the governor need to implement his plan, knowing that not all islanders are liars?

4. The rectangle $ABCD$ is partitioned into five rectangles P_1, P_2, P_3, P_4, P_5. If P_5 is a square, and P_1, P_2, P_3, P_4 have the same area, prove that $ABCD$ is a square.



5. Let M be a point on the semicircle with diameter AB, K be a point on AB, and P, Q be the circumcenters of triangles AMK, MKB. Prove that the points M, K, P, Q lie on a circle.

6. Three parabolas are defined by $y = f(x)$, $y = g(x)$, and $y = h(x)$, where f, g, h are quadratic polynomials. The branches of the parabolas are directed upwards. Let a, b, c be the x-coordinates of the vertices of the parabolas, respectively. If $f(b) < f(c)$ and $g(c) < g(a)$, prove that $h(b) < h(a)$.

7. Two towns A and B are connected by a straight road. The cyclists start a trip from A to B one after another at 8:00am, with equal time intervals between them, and move with equal and constant speeds. The motorcyclists start a trip from B to A one after another at 8:00am with equal time intervals, and move with equal and constant speeds. The first cyclist reaches B in 4:00pm, and in the same time the last motorcyclist reaches A. Let M be the midpoint of AB, and X, Y be the numbers of meetings of the cyclists and motorcyclists between 8:00 to 12:00, and between A and M, respectively. Assuming that there are no meetings at exactly 12:00, nor exactly at M, compare X and Y.

8. 65 beetles are placed in 65 cells of 9×9 table. In each move, every beetle creeps to an adjacent cell. No beetle makes two horizontal or two vertical moves in succession. Prove that after several moves at least two beetles will be in the same cell.

Category B

1. A point B inside a regular hexagon $A_1A_2\dots A_6$ is given, such that $\angle A_2A_1B = \angle A_4A_3B = 50^\circ$. Find $\angle A_1A_2B$.

2. Find the product of the distinct real numbers a,b,c, if
$$a^3 = 3(b^2+c^2)-25, \quad b^3 = 3(c^2+a^2)-25, \quad c^3 = 3(a^2+b^2)-25.$$

3. Words are formed with the letters A and B. Given words $x_1,x_2,\dots,x_n$, one can form the word $x_1x_2\dots x_n$ writing these words consequently. Prove that any word of length 1995 can be formed using less than 800 palindroms.

4. Find all functions $f:\mathbb{R}\to\mathbb{R}$ satisfying
$$f(f(x-y)) = f(x)-f(y)-f(x)f(y)-xy \quad \text{for all } x,y\in\mathbb{R}.$$

5. Let AK, BL, CM be the altitudes of an acute triangle ABC. If
$$9\overrightarrow{AK}+4\overrightarrow{BL}+7\overrightarrow{CM}=0,$$
prove that one of the angles of $\triangle ABC$ is equal to 60°.

6. Prove that among any three real numbers no two of which add up to 1, there are two numbers x,y such that $\dfrac{xy}{x+y-1}$ is not in the interval $(0,1)$.

7. Let $\mathbb{Q}_1$ be the set of all rational numbers greater than 1.
   (a) Is it possible to partition $\mathbb{Q}_1$ into (disjoint) sets A and B, so that the sum of any two numbers from one subset belongs to the same subset?
   (b) The same about products.

8. (a) Each side of an equilateral triangle is divided into 6 equal parts, and the partition points are connected by lines parallel to the sides of the triangle. Each vertex of the obtained triangle grid is occupied by exactly one beetle. All beetles begin creeping along the lines of the grid simultaneously with the same speed. They creep according to the following rule: whenever a beetle reaches a gridpoint, it turns to the left or right by 60° or 120°. Prove that at some moment two of them will meet at a gridpoint.
   (b) Would the statement remain true if each side of the triangle was divided into 5 equal parts instead?

Category A

1. There are 20 rooms in a hotel on a seaside. All the rooms are arranged along one side of a common corridor. The rooms are numbered 1 to 20 in this order. A visitor may rent only one room for two days, or two neighboring rooms for one day. The cost of a room is $1 per day. A sea-bathing seasons last 100 days. Room 1 is not rented the first day of a season, and room 20 is not rented the last day. Prove that the owners of the hotel receive at most $1996 per season.

2. (a) After a lesson in mathematics, the x-axis and the graph of the function $y = 2^x$ were left on the blackboard, but the y-axis and the scale were rubbed out. Give a Euclidean construction for the y-axis and the scale.

   (b) Give an Euclidean construction for the missing elements, if only the graph and a straight line parallel to the x-axis was left.

3. Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(f(x+y)) = f(x+y) + f(x)f(y) - xy \quad \text{for all } x,y \in \mathbb{R}.$$

4. In a triangle ABC with $\angle B = 3\angle A$, let M,N be chosen on side CA so that $\angle CBM = \angle MBN = \angle NBA$. Suppose that X is an arbitrary point on BC, L the intersection of AX and BN, and K the intersection of NX and BM. Prove that KL and AC are parallel.

5. The center O_1 of a circle S_1 lies on a circle S_2 with center O_2. The radius of S_2 is greater than that of S_1. Let A be the intersection of S_1 and O_1O_2. Consider a circle S centered at an arbitrary point X on S_2 and passing through A, and let $Y \neq A$ be the intersection of S and S_2. Prove that all lines XY are concurrent as X runs along S_2.

6. If the equation $x^3 + (a-1)\sqrt{3}x^2 - 6ax + b = 0$ has three real roots, prove that $|b| \le |a+1|^3$.

7. A lattice frame construction of a $2 \times 2 \times 2$ cube is formed with 54 metal shafts of unit length. An ant starts at some junction A, and creeps along the shafts by the following rule: whenever it reaches a junction, it turns to a perpendicular shaft, but it never visits the same junction twice. After some time the ant returns to the initial junction A. What is the maximum possible length of the ant's path?

8. Is it possible to partition the set of all rational numbers into two (disjoint) subsets A and B so that

   (a) the sum of any two numbers from the same subset belongs to A?

   (b) the sum of any two distinct numbers from the same subset belongs to A?