57-th Belarusian Mathematical Olympiad 2007

Final Round

Category C

First Day

1. Give quadrilateral $ABCD$ with $\angle CAD = 45^\circ$, $\angle ACD = 30^\circ$, $\angle BAC = \angle BCA = 15^\circ$, find the value of $\angle DBC$.

2. Prove the inequality
$$\frac{1}{x_1}+\frac{x_1}{x_2}+\frac{x_1x_2}{x_3}+\frac{x_1x_2x_3}{x_4}+\cdots+\frac{x_1x_2\cdots x_n}{x_{n+1}} \geq 4(1-x_1\cdots x_{n_1})$$
holds for all positive real numbers $x_1,\ldots,x_{n+1}$.

3. Each point of a circle is painted either black or white.
   (a) Prove that there exists an isosceles triangle inscribed in this circle such that all its vertices are painted the same color.
   (b) Does there exist an equilateral triangle inscribed in this circle such that all its vertices are painted the same color?

4. Given a $2n\times 2m$ table ($m,n\in\mathbb{N}$) with one of two signs ”+” or ”−” in each of its cells. A union of all the cells of some row and some column is called a *cross*. The cell on the intersectin of this row and this column is called the *center of the cross*. The following procedure is called a *transformation of the table*: we mark all cells which contain ”−” and then, in turn, we replace the signs in all cells of the crosses which centers are marked by the opposite signs, the signs in the centers of those crosses being unchanged. (It is easy to see that the order of the choice of the crosses doesn't matter.)

   Prove that any table can be obtained from some table applying such transformation one time.

Second Day

5. Find all triples of positive prime numbers p_1,p_2,p_3 such that
$$\begin{cases} 2p_1-p_2+7p_3 &= 1826,\\ 3p_1+5p_2+7p_3 &= 2007.\end{cases}$$

6. Let O be the point of intersection of the diagonals AC and BD of the quadrilateral $ABCD$ with $AB=BC$ and $CD=DA$. Let N and K be the feet of perpendiculars from D and B to ABD and BCD, respectively. Prove that the points N, O, and K are colinear.

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7. An equilateral triangle of side n ($n > 2$) is divided into n^2 equilateral triangles of side 1 with the lines parallel to the sides of the triangle. Can one label these unit triangles with numbers 1 up to n^2 such that the following two conditions are satisfied:

   (i) The triangles with the numbers i and $i+1$ have at least one common point for all $i = 1, 2, \dots, n^2 - 1$;

   (ii) The triangles with the numbers i and $i+2$ have at least one commont point for all $i = 1, 2, \dots, n^2 - 1$.

8. 2007 integers are arranged on a circle such that for any five successive numbers the sum of some three of them is twice the some of the other two. Prove that all these numbers have to be zero.

Category B

First Day

1. Prove that the number

$$\sqrt{8.\underbrace{000\dots01}_{n}}$$

   is irrational for every positive integer n.

2. Given a $2n \times 2m$ table ($m, n \in \mathbb{N}$) with one of two signs "$+$" or "$-$" in each of its cells. A union of all the cells of some row and some column we call a *cross*. The cell on the intersection of this row and this column is called a *center of the cross*. The following procedure we call a *transformation of the table*: we mark all cells which contain "$-$" and then, in turn, we replace the signs in all cells of the crosses which centers are marked by the opposite signs. (It is easy to see that the order of the choice of the crosses doesn't matter.)

   Find all arrangements of the signs in the table such that two consecutive transformations of the table give the table with the signs "$+$" only.

3. Three beetles are at the same point of the table. Suddenly they begin to crawl and after a while they are at the vertices of a triangle with the inradius equal to 2

   Prove that at least on of the beetles crawls the distance which is greater than 3.

4. Each point of a circle is painted either black or white.

   (a) Prove that there exists a quadrilateral inscribed in this circle, with two parallel sides (i.e. a trapezoid or a rectangle), such that all its vertices are painted the same color.

   (b) Can one one claim that there exists an inscribed *rectangle* such that all its vertices are painted the same color.

   (c) Can one claim that there exists an inscribed *trapezoid* such that all its vertices are painted the same color?

Second Day

5. Given a convex quadrilateral $ABCD$ with $\angle ACB = \angle ADB$, $AB = AD$. Let N and K be the feet of perpendiculars from A onto the lines CB and DB, respectively. Prove that $NK \perp AC$.

6. Find all positive integers k with the following property: There are four distinct divisors k_1, k_2, k_3, k_4 of k such that k divides $k_1 + k_2 + k_3 + k_4$.

7. Real numbers a, b, c, and d satisfy the inequality
$$abcd > a^2 + b^2 + c^2 + d^2.$$
Prove that $abcd > a + b + c + d + 8$.

8. Let (m, n) be a pair of positive integers.
   (a) Prove that the set of all positive integers can be partitioned into three pairwise disjoint nonempty subsets such that none of them has two numbers with absolute value of their difference equal to either m or n.
   (b) Find all pairs (m, n) such that the set of all positive integers can not be partitioned into two pairwise disjoint nonempty subsets satisfying the above condition.

Category A

First Day

1. Find all polynomials $P(x)$ of degree $\leq n$ with nonnegative real coefficients such that the inequality $P(x)P(1/x) \leq (P(1))^2$ holds for all positive real numbers x.

2. Circles S_1 and S_2 with centers O_1 and O_2, respectively, pass through the centers of each other. Let A be one of their intersection points. Two points M_1 and M_2 begin to move simultaneously starting from A. Point M_1 moves along S_1 and point M_2 moves along S_2. Both points move in clockwise direction and have the same linear velocity v.
   (a) Prove that all triangles AM_1M_2 are equilateral.
   (b) Determine the trajectory of the movement of the center of the triangle AM_1M_2 and find its linear velocity.

3. Given a $2n \times 2m$ table ($m, n \in \mathbb{N}$) with one of two signs "$+$" or "$-$" in each of its cells. A union of all the cells of some row and some column is called a *cross*. The cell on the intersectin of this row and this column is called the *center of the cross*. The following procedure we call a *transformation of the table*: we mark all cells which contain "$-$" and then, in turn, we replace the signs in all cells of the crosses which centers are marked by the opposite signs. (It is easy to see that

the order of the choice of the crosses doesn't matter.) We call a table *attainable* if it can be obtained from some table applying such transformations one time.

Find the number of all attainable tables.

4. Each point of a circle is painted in one of the N colors ($N \geq 2$). Prove that there exists an inscribed trapezoid such that all its vertices are painted the same color.

Second Day

5. Let O be the intersection point of the diagonals of the convex quadrilateral $ABCD$, $AO = CO$. Points P and Q are marked on the segments AO and CO, respectively, such that $PO = OQ$. Let N and K be the intersection points of the sides AB, CD, and the lines DP and BQ respectively. Prove that the points N, O, and K are colinear.

6. Let a be the sum and b the product of the real roots of the equation $x^4 - x^3 - 1 = 0$. Prove that $b < -11/10$ and $a > 6/11$.

7. Find all positive integers n and m satisfying the equality

$$n^5 + n^4 = 7^m - 1.$$

8. Let (m, n) be a pair of positive integers.

   (a) Prove that the set of all positive integers can be partitioned into four pairwise disjoint nonempty subsets such that none of them has two numbers with absolute value of their difference equal to either m, n, or $m + n$.

   (b) Find all pairs (m, n) such that the set of all positive integers can not be partitioned into three pairwise disjoint nonempty subsets satisfying the above condition.

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