54-th Belarusian Mathematical Olympiad 2004

Final Round

Category C

First Day

1. An $n \times n$ square table ($n \geq 3$) is filled with integers such that the sum of numbers in any 2×2 or 3×3 square is even. Find all n for which the sum of all numbers in the table is necessarily even.

2. The sequence of real numbers (a_n) satisfies the relation $a_{n+1} = a_n(a_n + 2)$ for all $n \in \mathbb{N}$. Find the set of possible values of a_{2004}.

3. Find all pairs of integers (x, y) satisfying the equation
$$y^2(x^2 + y^2 - 2xy - x - y) = (x + y)^2(x - y).$$

4. Let K, L, M, N be the midpoints of the sides AB, BC, CD, DA of a convex quadrilateral $ABCD$, respectively, and let NL and KM intersect at T. Prove that
$$\frac{8}{3}S_{DNTM} < S_{ABCD} < 8S_{DNTM}.$$

Second Day

5. The Mathematical Olympiad paper consists of 8 problems. We say that a contestant is *strong* if he successfully solved more than four problems. A problem is said to be *difficult* if it is solved by less than a half of all strong contestants.
   (a) Find the largest possible number of difficult problems on the olympiad.
   (b) Assuming that there were the largest possible number of difficult problems, find the smallest possible even number and the smallest possible odd number of strong contestants.

6. Circles S_1 and S_2 meet at points A and B. A line through A is parallel to the line through the centers of S_1 and S_2 and meets S_1 again at C and S_2 again at D. The circle S_3 with diameter CD meets S_1 and S_2 again at P and Q, respectively. Prove that lines CP, DQ, and AB are concurrent.

7. Let be given two similar triangles such that the altitudes of the first triangle are equal to the sides of the other. Find the largest possible value of the similarity ratio of the triangles.

8. A sequence a of k digits is called *stable* if the product of any two numbers ending with a in the decimal system also ends with a. For instance, 0 and 25 are stable sequences. Prove that for any k there are exactly four stable k-digit sequences.

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Category B

First Day

1. The diagonals AD, BE, CF of a convex hexagon $ABCDEF$ meet at point O. Find the smallest possible area of this hexagon, if the areas of the triangles AOB, COD, EOF are equal to 4, 6, and 9, respectively.

2. An $n \times n$ square table ($n \geq 3$) is filled with integers such that the sum of numbers in any 3×3 or 5×5 square is even. Find all n for which the sum of all numbers in the table is necessarily even.

3. Let $a_0, a_1, \dots, a_n$ be integers not less than -1 and not all equal to zero. Prove that if $a_0 + 2a_1 + 2^2 a_2 + \cdots + 2^n a_n = 0$, then $a_0 + a_1 + \cdots + a_n > 0$.

4. Consider the equation $c(ac+1)^2 = (5c+2b)(2c+b)$, where a, b, c are integers.

   (a) Prove that if c is odd, then it is a perfect square.

   (b) Is there a solution (in integers) with c even?

   (c) Prove that the equation has infinitely many integer solutions.

Second Day

5. Let $ABCD$ be a cyclic quadrilateral with $AB \cdot BC = 2AD \cdot DC$. Prove that the diagonals AC and BD satisfy $8BD^2 \leq 9AC^2$.

6. (a) Suppose that there is a point X in the plane of a given quadrilateral $ABCD$ such that the perimeters of triangles ABX, BCX, CDX, DAX are equal. Show that $ABCD$ is a cyclic quadrilateral.

   (b) If $ABCD$ is cyclic, does there necessarily exist a point X such that the perimeters of triangles ABX, BCX, CDX, DAX are equal?

7. A teacher wrote $n > 2$ positive integers on a blackboard such that none of them divides any other. Students alternately erase the numbers one by one, under the condition that a student can only erase a number if it divides the sum of numbers that are on the blackoard at that moment. Is it possible for every $n > 2$ that exactly two numbers remain on the blackboard?

8. At a mathematical olympiad, eight problems were given to 30 contestants. In order to take the difficulty of each problem into account, the jury decided to assign weights to the problems as follows: a problem is worth n points if it was not solved by exactly n contestants. For example, if a problem was solved by all contestants, then it is worth no points. (It is assumed that there are no partial marks for a problem.)

   (a) Is it possible that a student solves more problems than anybody else, but gains less points than anybody else?

(b) Is it possible that a student solves less problems than anybody else, but gains more points than anybody else?

Category A

First Day

1. A connected graph with at least one vertex of an odd degree is given. Show that one can color the edges of the graph red and blue in such a way that, for each vertex, the absolute difference between the numbers of red and blue edges at that vertex does not exceed 1.

2. Let C be a semicircle with diameter AB. Circles S, S_1, S_2 with radii r, r_1, r_2, respectively, are tangent to C and the segment AB, and moreover S_1 and S_2 are externally tangent to S. Prove that $\frac{1}{\sqrt{r_1}} + \frac{1}{\sqrt{r_2}} = \frac{2\sqrt{2}}{\sqrt{r}}$.

3. The cells of an $n \times n$ table ($n \geq 3$) are painted black and white in the chess-like manner. Per move one can choose any 2×2 square and reverse the color of the cells inside it. Find all n for which one can obtain a table with all cells of the same color after finitely many such moves.

4. For a positive integer $A = \overline{a_n \dots a_1 a_0}$ with nonzero digits which are not all the same ($n \geq 0$), the numbers $A_k = \overline{a_{n-k} \dots a_1 a_0 a_n \dots a_{n-k+1}}$ are obtained for $k = 1, 2, \dots, n$ by cyclic permutations of its digits. Find all A for which each of the A_k is divisible by A.

Second Day

5. Suppose that A and B are sets of real numbers such that

$$A \subset B + \alpha\mathbb{Z} \quad \text{and} \quad B \subset A + \alpha\mathbb{Z} \quad \text{for all } \alpha > 0.$$

(where $X + \alpha\mathbb{Z} = \{x + \alpha n \mid x \in X,\, n \in \mathbb{Z}\}$).

(a) Does it follow that $A = B$?

(b) The same question, with the assumption that B is bounded.

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Ivan got less points than any other contestant. Find the greatest score he can have.

7. A cube $ABCDA_1B_1C_1D_1$ is given. Find the locus of points E on the face $A_1B_1C_1D_1$ for which there exists a line intersecting the lines AB, A_1D_1, B_1D, and EC.

8. Tom Sawer must whitewash a circular fence consisting of N planks. He whitewashes the fence going clockwise and following the rule: He whitewashes the first plank, skips two planks, whitwashes one, skips three, and so on. Some planks may be whitewashed several times. Tom believes that all planks will be whitewashed sooner or later, but aunt Polly is sure that some planks will remain unwhitewashed forever. Prove that Tom is right if N is a power of two, otherwise aunt Polly is right.

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