4-th Bosnia and Hercegovina Mathematical Olympiad 1999

Sarajevo, May 22–23, 1999

First Day

1. Let a, b, c be side lengths of a triangle. Prove that at least one of the equations
$$x^2 - 2bx + 2ac = 0, \quad x^2 - 2cx + 2ab, \quad x^2 - 2ax + 2bc$$
has no real solutions.

2. If a, b, c are the sides and R the circumradius of a triangle ABC, prove that
$$\frac{a^2}{b+c-a} + \frac{b^2}{c+a-b} + \frac{c^2}{a+b-c} \geq 3R\sqrt{3}.$$

3. Let $f : [0,1] \to \mathbb{R}$ be an injective function with $f(0) + f(1) = 1$. Show that there exist $x_1, x_2 \in [0,1]$ such that $2f(x_1) < f(x_2) + \frac{1}{2}$. Can you generalize this result?

Second Day

4. In a triangle ABC, the angle bisectors of the angles at A and B meet the opposite sides at D and E, respectively. Let F and G be the projections of C onto the lines AD and BE. Prove that FG is parallel to AB.

5. For a nonempty set S, let $\sigma(S)$ and $\pi(S)$ denote the sum and product of elements of S, respectively. Prove that
   (a) $\sum \frac{1}{\pi(S)} = n$;
   (b) $\sum \frac{\sigma(S)}{\pi(S)} = (n^2 + 2n) - (n+1)\left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right)$,

   where the sums extend over all nonempty subsets S of $\{1, 2, \ldots, n\}$.

6. Consider the polynomial $P(x) = x^4 + 3x^3 + 3x + p$, where p is a real number.
   (a) Find p such that $P(x)$ has an imaginary root x_1 with $|x_1| = 1$ and $2Re(x_1) = \frac{1}{2}\left(\sqrt{17} - 3\right)$.
   (b) For this value of p, find all other roots of $P(x)$.
   (c) Show that there is no $n \in \mathbb{N}$ for which $x_1^n = 1$.

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