2-nd Bosnia and Hercegovina Mathematical Olympiad 1997

Sarajevo, May 1997

First Day

1. Find all real solutions of the system of equations:

$$\begin{aligned} 8(x^3+y^3+z^3) &= 73, \\ 2(x^2+y^2+z^2) &= 3(xy+yz+zx), \\ xyz &= 1. \end{aligned}$$

2. In an isosceles triangle ABC with the base AB, O is the circumcenter and S the incenter. Point M is chosen on side BC. Prove that $SM \parallel AC$ if and only if $OM \perp BS$.

3. Let A be a subset of $\mathbb{R}$. A function $f : A \to \mathbb{R}$ satisfies the condition

$$f(x+y) = f(x)f(y) - f(xy) + 1 \quad \text{for all } x, y \in A.$$

   (i) If $A \supseteq \mathbb{N}$ and $c = f(1) - 1$, show that for all n

$$f(n) = \begin{cases} \dfrac{c^{n+1}-1}{c-1} & \text{if } c \neq 1, \\ n+1 & \text{if } c = 1. \end{cases}$$

   (ii) Find all such functions if $A = \mathbb{N}$.

   (iii) If $A = \mathbb{Q}$, find all such functions with $f(1997) \neq f(1998)$.

Second Day

4. (a) The incircle of a triangle ABC is tangent to the sides BC, CA, AB at A_1, B_1, C_1, respectively. Let B_1C_1, C_1A_1, A_1B_1 be the arcs not containing A_1, B_1, C_1, respectively, and let I_1, I_2, I_3 be their respective arc lengths. If a, b, c denote the side lengths of triangle ABC, prove that

$$\frac{a}{I_1} + \frac{b}{I_2} + \frac{c}{I_3} \geq \frac{9\sqrt{3}}{\pi}.$$

   (b) Let $ABCD$ be a tetrahedron with $AB = CD = a$, $BC = AD = b$, $AC = BD = c$. Express the heights of the tetrahedron in terms of a, b and c.

5. (a) Show that for every positive integer n there exists a set M_n consisting of n positive integers and having the following property:

   (1) the arithmetic mean of elements of any nonempty subset of M is an integer

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    (2) the geometric mean of elements of any nonempty subset of M is an integer

    (3) both arithmetic and geometric mean of elements of any nonempty subset of M are integers.

  (b) Is there an infinite set M of positive integers having property (1)?

6. Let k, m, n be integers with $1 < n \leq m - 1 \leq k$. Determine the maximum size of subsets S of the set $\{1, 2, \dots, k\}$ such that no sum of n distinct elements of S is

  (a) equal to m;

  (b) bigger than m.