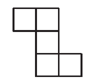
11-th Bosnia and Hercegovina Mathematical Olympiad 2006

Sarajevo, May 20–21, 2006

First Day

1. A *Z-figure* is a figure congruent to the figure shown. At least, how many *Z*-figures are needed to cover an 8×8 chessboard? (The *Z*-figures may overlap.)



2. A triangle ABC is given. Determine the locus of the centers of rectangles inscribed in triangle ABC with one side lying on side AB.

3. Prove that for every positive integer n it holds that $\{n\sqrt{7}\} > \dfrac{3\sqrt{7}}{14n}$, where $\{x\}$ is the fractional part of x.

Second Day

4. Prove that every infinite arithmetic sequence $a, a+d, a+2d, \dots$, where $a, d \in \mathbb{N}$, contains an infinite geometric subsequence $b, bq, bq^2, \dots$, where $b, q \in \mathbb{N}$.

5. An acute-angled triangle ABC is inscribed in a circle with center O. A point P is taken on the shorter arc AB. The perpendicular from P to BO intersects AB at S and BC at T. Likewise, the perpendicular from P to AO intersects AB at Q and AC at R.

   (a) Prove that the triangle PQS is isosceles.

   (b) Show that $PQ^2 = QR \cdot ST$.

6. Let $a_1, a_2, \dots, a_n$ be real constants and

$$f(x) = \cos(a_1 + x) + \frac{\cos(a_2 + x)}{2} + \frac{\cos(a_3 + x)}{2^2} + \cdots + \frac{\cos(a_n + x)}{2^{n-1}}.$$

   If x_1, x_2 are real and $f(x_1) = f(x_2) = 0$, prove that $x_1 - x_2 = m\pi$ for some integer m.