10-th Bosnia and Hercegovina Mathematical Olympiad 2005

Banja Luka, May 7–8, 2005

First Day

1. Point H is the orthocenter of a triangle ABC. Prove that the midpoints of the segments AB and CH and the intersection point of the bisectors of angles $\angle CAH$ and $\angle CBH$ are collinear.

2. If a_1, a_2, a_3 are nonnegative real numbers with $a_1 + a_2 + a_3 = 1$, prove the inequality
$$a_1\sqrt{a_2} + a_2\sqrt{a_3} + a_3\sqrt{a_1} \le \frac{1}{\sqrt{3}}.$$

3. An integer $n \ge 2$ is given. Let $x_1, x_2, \dots, x_n$ be distinct positive integers and let S_i be the sum of these numbers with x_i excluded, $i = 1, 2, \dots, n$. Define
$$f(x_1, x_2, \dots, x_n) = \frac{\gcd(x_1, S_1) + \gcd(x_2, S_2) + \cdots + \gcd(x_n, S_n)}{x_1 + x_2 + \cdots + x_n}.$$
Find the largest value of f over all possible n-tuples $(x_1, \dots, x_n)$.

Second Day

4. Point A is chosen on the line containing a diameter PQ of the circle $k(S, r)$, outside the circle. A tangent t to k passes through A and meets the circle at point T. Let p and q denote the tangents to k at P and Q, respectively, and let $PT \cap q = \{N\}$ and $QT \cap p = \{M\}$. Show that the points A, M, N are collinear.

5. Suppose that a permutation $(a_1, a_2, \dots, a_n)$ of numbers $1, 2, \dots, n$ satisfies
$$\frac{a_k^2}{a_{k+1}} \le k + 2 \quad \text{for } k = 1, 2, \dots, n-1.$$
Prove that it must be the identity permutation.

6. Prove that if integers a, b, c satisfy $\dfrac{a}{b} + \dfrac{b}{c} + \dfrac{c}{a} = 3$, then abc is a perfect cube.

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