7-th Bosnia and Hercegovina Mathematical Olympiad 2002

May 2002

First Day

1. Let x, y, z be real numbers that satisfy
$$x+y+z=3 \quad \text{and} \quad xy+yz+zx=a,$$
where a is a real parameter. Find the value of a for which the difference between the maximum and minimum possible values of x equals 8.

2. Triangle ABC is given in a plane. Draw the line that connects the points where the bisectors of angles ABC and ACB meet the opposite sides of the triangle. Through the point of intersection of this line and the bisector of angle BAC, draw a line parallel to BC. Let this line intersect AB in M and AC in N. Prove that $2MN = BM + CN$.

3. If n is a natural number, show that $(n+1)(n+2)\cdots(n+10)$ is not a perfect square.

Second Day

4. Real numbers a, b, c satisfy $a^2+b^2+c^2=1$. Prove the inequality
$$\frac{a^2}{1+2bc}+\frac{b^2}{1+2ca}+\frac{c^2}{1+2ab} \geq \frac{3}{5}.$$

5. Let p and q be different prime numbers. Solve the following system in integers:
$$\begin{aligned} \frac{z+p}{x}+\frac{z-p}{y} &= q, \\ \frac{z+p}{y}-\frac{z-p}{x} &= q. \end{aligned}$$

6. The vertices of the convex quadrilateral $ABCD$ and the intersection point S of its diagonals are integer points in the plane. Let P be the area of $ABCD$ and P_1 the area of triangle ABS. Prove that
$$\sqrt{P} \geq \sqrt{P_1}+\frac{\sqrt{2}}{2}.$$

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