25-th Balkan Mathematical Olympiad

Ohrid, FYR Macedonia – May 6, 2008

1. An acute-angled scalene triangle ABC with $AB > BC$ is given. Let O be its circumcenter, H its orthocenter, and F the foot of the altitude from C. Let P be the point (other than A) on the line AB such that $AF = PF$, and M be a point on AC. We denote the intersection of PH and BC by X, the intersection of OM and FX by Y, and the intersection of OF and AC by Z. Prove that the points F, M, Y, and Z are concyclic.

2. Does there exist a sequence $a_1, a_2, \ldots$ of positive numbers satisfying both of the following conditions:

   (i) $\sum_{i=1}^{n} a_i \leq n^2$ for every positive integer n;

   (ii) $\sum_{i=1}^{n} \frac{1}{a_i} \leq 2008$ for every positive integer n?

3. Let n be a positive integer. The rectangle $ABCD$ with side lengths $90n+1$ and $90n+5$ is partitioned into unit squares with sides parallel to the sides of $ABCD$. Let S be the set of all points which are vertices of theses unit squares. Prove that the number of lines which pass though at least two points from S is divisible by 4.

4. Let c be a positive integer. The sequence $a_1, a_2, \ldots$ is defined by $a_1 = c$ and $a_{n+1} = a_n^2 + a_n + c^3$ for every positive integer n. Find all values of c for which there exists some integers $k \geq 1$ and $m \geq 2$ such that $a_k^2 + c^3$ is the m-th power of some positive integer.

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