18-th Balkan Mathematical Olympiad

Belgrade, Yugoslavia – May 5, 2001

1. Let n be a positive integer. Prove that if a,b are integers greater than 1 such that $ab = 2^n - 1$, then the number $ab - (a-b) - 1$ is of the form $k \cdot 2^{2m}$, where k is odd and m a positive integer.

2. Prove that a convex pentagon that satisfies the following two conditions must be regular:

   (i) All its interior angles are equal;

   (ii) The lengths of all its sides are rational numbers.

3. Let a,b,c be positive real numbers such that $a+b+c \geq abc$. Prove that
$$a^2+b^2+c^2 \geq abc\sqrt{3}.$$

4. A cube of edge 3 is divided into 27 unit cube cells. One of these cells is empty, while in the other cells there are unit cubes which are arbitrarily denoted by $1,2,\dots,26$. An *legal* move consists of moving a unit cube into a neighboring empty cell (two cells are neighboring if they share a face). Does there exist a finite sequence of legal moves after which any two cubes denoted by k and $27-k$ ($k = 1,2,\dots,13$) will exchange their positions?

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