16-th Balkan Mathematical Olympiad

Ohrid, Macedonia – May 8, 1999

1. Let D be the midpoint of the shorter arc BC of the circumcircle of an acute-angled triangle ABC. The points symmetric to D with respect to BC and the circumcenter are denoted by E and F, respectively. Let K be the midpoint of EA.

   (a) Prove that the circle passing through the midpoints of the sides of $\triangle ABC$ also passes through K.

   (b) The line through K and the midpoint of BC is perpendicular to AF.

2. Let $p > 2$ be a prime number with $3 \mid p-2$. Consider the set

$$S = \{y^2 - x^3 - 1 \mid x, y \in \mathbb{Z},\ 0 \le x, y \le p-1\}.$$

   Prove that at most $p-1$ elements of S are divisible by p.

3. Let M, N, P be the orthogonal projections of the centroid G of an acute-angled triangle ABC onto AB, BC, CA, respectively. Prove that

$$\frac{4}{27} < \frac{S_{MNP}}{S_{ABC}} \le \frac{1}{4}.$$

4. Let $0 \le x_0 \le x_1 \le x_2 \le \cdots$ be a sequence of nonnegative integers such that for every $k \ge 0$ the number of terms of the sequence which do not exceed k is finite, say y_k. Prove that for all positive integers m, n,

$$\sum_{i=0}^{n} x_i + \sum_{j=0}^{m} y_j \ge (n+1)(m+1).$$

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com