14-th Balkan Mathematical Olympiad

Kalabaka, Greece – April 30, 1997

1. Suppose that O is a point inside a convex quadrilateral $ABCD$ such that
$$OA^2 + OB^2 + OC^2 + OD^2 = 2S_{ABCD},$$
where S_{ABCD} denotes the area of $ABCD$. Prove that $ABCD$ is a square and O its center. *(Yugoslavia)*

2. Let $\mathscr{A} = \{A_1, A_2, \dots, A_k\}$ be a collection of subsets of an n-element set S. If for any two elements $x, y \in S$ there is a subset $A_i \in \mathscr{A}$ containing exactly one of the two elements x, y, prove that $2^k \geq n$. *(Yugoslavia)*

3. Circles C_1 and C_2 touch each other externally at D, and touch a circle Γ internally at B and C, respectively. Let A be an intersection point of Γ and the common tangent to C_1 and C_2 at D. Lines AB and AC meet C_1 and C_2 again at K and L, respectively, and the line BC meets C_1 again at M and C_2 again at N. Show that the lines AD, KM, LN are concurrent. *(Greece)*

4. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy
$$f(xf(x) + f(y)) = f(x)^2 + y \quad \text{for all } x, y.$$
(Bulgaria)

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com