13-th Balkan Mathematical Olympiad

Bacau, Romania – April 30, 1996

1. Let O be the circumcenter and G be the centroid of a triangle ABC. If R and r are the circumcenter and incenter of the triangle, respectively, prove that

$$OG \leq \sqrt{R(R-2r)}.$$

(Greece)

2. Let $p > 5$ be a prime. Consider $X = \{p - n^2 \mid n \in \mathbb{N}\}$. Prove that there are two distinct elements $x, y \in X$ such that $x \neq 1$ and $x \mid y$. *(Albania)*

3. In a convex pentagon $ABCDE$, M, N, P, Q, R are the midpoints of the sides AB, BC, CD, DE, EA, respectively. If the segments AP, BQ, CR, DM pass through a single point, prove that EN contains that point as well. *(Yugoslavia)*

4. Show that there exists a subset A of the set $\{1, 2, \ldots, 2^{1996} - 1\}$ with the following properties:

   (i) $1 \in A$ and $2^{1996} - 1 \in A$;

   (ii) Every element of $A \setminus \{1\}$ is the sum of two (possibly equal) elements of A;

   (iii) A contains at most 2012 elements. *(Romania)*