3-rd Balkan Mathematical Olympiad

Bucharest, Romania – May 5-10, 1986

1. A line through the incenter I of a triangle ABC intersects its circumcircle at F and G, and its incircle at D and E, where D is between I and F. Prove that $DF \cdot EG \geq r^2$, where r is the inradius. When does equality occur? *(Greece)*

2. Let E,F,G,H,K,L respectively be points on the edges AB,BC,CA,DA, DB,DC of a tetrahedron $ABCD$. If

$$AE \cdot BE = BF \cdot CF = CG \cdot AG = DH \cdot AH = DK \cdot BK = DL \cdot CL,$$

prove that the points E,F,G,H,K,L lie on a sphere. *(Bulgaria)*

3. The sequence (a_n) is defined by

$$a_1 = a, \quad a_2 = b, \quad \text{and} \quad a_{n+1} = \frac{a_n^2 + c}{a_{n-1}} \quad \text{for } n \geq 2,$$

where a,b,c are real numbers with $ab \neq 0$, $c > 0$. Prove that all terms of this sequence are integers if and only if a,b and $\dfrac{a^2+b^2+c}{ab}$ are integers.

(Romania)

4. A triangle ABC and a point T are given in the plane so that the triangles TAB,TBC,TCA have the same area and perimeter. Prove that:

   (i) If T is inside $\triangle ABC$, then $\triangle ABC$ is equilateral;

   (ii) If T is not inside $\triangle ABC$, then $\triangle ABC$ is right-angled. *(Romania)*

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