19-th Austrian Mathematical Olympiad 1988

Final Round

First Day - June 8

1. If $a_1, \dots, a_{1988}$ are positive numbers whose arithmetic mean is 1988, show that
$$\sqrt[1988]{\prod_{i,j=1}^{1988}\left(1+\frac{a_i}{a_j}\right)} \geq 2^{1988}$$
and determine when equality holds.

2. An equilateral triangle $A_1A_2A_3$ is divided into four smaller equilateral triangles by joining the midpoints A_4, A_5, A_6 of its sides. Let $A_7, \dots, A_{15}$ be the midpoints of the sides of these smaller triangles. The 15 points $A_1, \dots, A_{15}$ are each colored either green or blue. Show that with any such colouring there are always three mutually equidistant points A_i, A_j, A_k having the same color.

3. Show that there is precisely one sequence $a_1, a_2, \dots$ of integers which satisfies $a_1 = 1$, $a_2 > 1$, and
$$a_{n+1}^3 + 1 = a_n a_{n+2} \quad \text{for } n \geq 1.$$

Second Day - June 9

4. Let a_{ij} be nonnegative integers such that $a_{ij} = 0$ if and only if $i > j$ and that $\sum_{j=1}^{1988} a_{ij} = 1988$ holds for all $i = 1, \dots, 1988$. Find all real solutions of the system of equations
$$\sum_{j=1}^{1988}(1+a_{ij})x_j = i+1, \quad 1 \leq i \leq 1988.$$

5. The bisectors of angles B and C of triangle ABC intersect the opposite sides in points B' and C' respectively. Show that the line $B'C'$ intersects the incircle of the triangle.

6. Determine all monic polynomials $p(x)$ of fifth degree having real coefficients and the following property: Whenever a is a (real or complex) root of $p(x)$, then so are $1/a$ and $1-a$.

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com