18-th Austrian Mathematical Olympiad 1987

Final Round

First Day - June 2

1. The sides a, b and the bisector of the included angle γ of a triangle are given. Determine necessary and sufficient conditions for such triangles to be constructible and show how to reconstruct the triangle.

2. Find the number of all sequences $(x_1,\dots,x_n)$ of letters a,b,c that satisfy $x_1 = x_n = a$ and $x_i \neq x_{i+1}$ for $1 \le i \le n-1$.

3. Let $x_1,\dots,x_n$ be positive real numbers. Prove that
$$\sum_{k=1}^{n} x_k + \sqrt{\sum_{k=1}^{n} x_k^2} \le \frac{n+\sqrt{n}}{n^2}\left(\sum_{k=1}^{n}\frac{1}{x_k}\right)\left(\sum_{k=1}^{n} x_k^2\right).$$

Second Day - June 3

4. Find all triples (x,y,z) of natural numbers satisfying $2xz = y^2$ and $x+z = 1987$.

5. Let P be a point in the interior of a convex n-gon $A_1A_2\dots A_n$ ($n \ge 3$). Show that among the angles $\beta_{ij} = \angle A_iPA_j$, $1 \le i \le n$, there are at least $n-1$ angles satisfying $90^\circ \le \beta_{ij} \le 180^\circ$.

6. Determine all polynomials $P_n(x) = x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$ with integer coefficients whose n zeros are precisely the numbers $a_1,\dots,a_n$ (counted with their respective multiplicities).

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