14-th Austrian Mathematical Olympiad 1983

Final Round

First Day

1. For every natural number x, let $Q(x)$ be the sum and $P(x)$ the product of the (decimal) digits of x. Show that for each $n \in \mathbb{N}$ there exist infinitely many values of x such that
$$Q(Q(x)) + P(Q(x)) + Q(P(x)) + P(P(x)) = n.$$

2. Let x_1, x_2, x_3 be the roots of $x^3 - 6x^2 + ax + a = 0$. Find all real numbers a for which $(x_1 - 1)^3 + (x_2 - 1)^3 + (x_3 - 1)^3 = 0$. Also, for each such a, determine the corresponding values of x_1, x_2, and x_3.

3. Let P be a point in the plane of a triangle ABC. Lines AP, BP, CP respectively meet lines BC, CA, AB at points A', B', C'. Points A'', B'', C'' are symmetric to A, B, C with respect to A', B', C', respectively. Show that
$$S_{A''B''C''} = 3S_{ABC} + 4S_{A'B'C'}.$$

Second Day

4. The sequence $(x_n)_{n \in \mathbb{N}}$ is defined by $x_1 = 2$, $x_2 = 3$, and
$$\begin{aligned} x_{2m+1} &= x_{2m} + x_{2m-1} && \text{for } m \geq 1; \\ x_{2m} &= x_{2m-1} + 2x_{2m-2} && \text{for } m \geq 2. \end{aligned}$$
Determine x_n as a function of n.

5. Given positive integers a, b, find all positive integers x, y satisfying the equation $x^{a+b} + y = x^a y^b$.

6. Planes π_1 and π_2 in Euclidean space $\mathbb{R}^3$ partition $S = \mathbb{R}^3 \setminus (\pi_1 \cup \pi_2)$ into several components. Show that for any cube in $\mathbb{R}^3$, at least one of the components of S meets at least three faces of the cube.

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