# 40-th Austrian Mathematical Olympiad 2009
## Final Round

### Part 1 – May 22

1. Show that for all positive integer $n$ the following inequality holds:
$$3^{n^2} > (n!)^4.$$

2. For a positive integers $n,k$ we define $k$-multifactorial of $n$ as $F_k(n) = n \cdot (n-k) \cdot (n-2k) \cdots r$, where $r$ is the reminder when $n$ is divided by $k$ that satisfy $1 \le r \le k$. Determine all non-negative integers $n$ such that $F_{20}(n) + 2009$ is a perfect square.

3. There are $n$ bus stops placed around the circular lake. Each bus stop is connected by a road to the two adjacent stops (we call a *segment* the entire road between two stops). Determine the number of bus routes that start and end in the fixed bus stop $A$, pass through each bus stop at least once and travel through exactly $n+1$ segments.

4. Let $D$, $E$, and $F$ be respectively the midpoints of the sides $BC$, $CA$, and $AB$ of $\triangle ABC$. Let $H_a$, $H_b$, $H_c$ be the feet of perpendiculars from $A$, $B$, $C$ to the opposite sides, respectively. Let $P$, $Q$, $R$ be the midpoints of the $H_bH_c$, $H_cH_a$, and $H_aH_b$ respectively. Prove that $PD$, $QE$, and $RF$ are concurrent.

### Part 2 – June 10–June 11

*First Day*

1. If $x,y,k,m \in \mathbb{N}$ let us define:
$$\alpha_m = \underbrace{2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}}_{m}, \quad A_{km}(x) = \underbrace{2^{2^{\cdot^{\cdot^{\cdot^{2^{x^{\alpha_m}}}}}}}}_{k \text{ twos}}, \quad B_k(y) = \underbrace{4^{4^{4^{\cdot^{\cdot^{\cdot^{4^{y}}}}}}}}_{m \text{ fours}}.$$
Determine all pairs $(x,y)$ of non-negative integers, dependent on $k>0$, such that $A_{kk}(x) = B_k(y)$.

2. (a) For positive integers $a<b$ let
$$M(a,b) = \frac{\sum_{k=a}^{b} \sqrt{k^2+3k+3}}{b-a+a}.$$
Calculate $[M(a,b)]$.

   (b) Calculate
$$N(a,b) = \frac{\sum_{k=a}^{b} \left[\sqrt{k^2+3k+3}\right]}{b-a+1}.$$

3. Let $P$ be the point in the interior of $\triangle ABC$. Let $D$ be the intersection of the lines $AP$ and $BC$ and let $A'$ be the point such that $\overrightarrow{AD} = \overrightarrow{DA'}$. The points $B'$ and $C'$ are defined in the similar way. Determine all points $P$ for which the triangles $A'BC$, $AB'C$, and $ABC'$ are congruent to $\triangle ABC$.

*Second Day*

4. Let $a$ be a positive integer. Consider the sequence $(a_n)$ defined as $a_0 = a$ and $a_n = a_{n-1} + 40^{n!}$ for $n > 0$. Prove that the sequence $(a_n)$ has infinitely many numbers divisible by 2009.

5. Let $n > 1$ and for $1 \leq k \leq n$ let $p_k = p_k(a_1, a_2, \dots, a_n)$ be the sum of the products of all possible combinations of $k$ of the numbers $a_1$, $a_2$, $\dots$, $a_n$. Furthermore let $P = P(a_1, a_2, \dots, a_n)$ be the sum of all $p_k$ with odd values of $k$ less than or equal to $n$.

   How many different values are taken by $a_j$ if all the numbers $a_j$ ($1 \leq j \leq n$) and $P$ are prime?

6. The quadrilateral $PQRS$ whose vertices are the midpoints of the sides $AB$, $BC$, $CD$, $DA$, respectively of a quadrilateral $ABCD$ is called the *midpoint quadrilateral* of $ABCD$.

   Determine all circumscribed quadrilaterals whose mid-point quadrilaterals are squares.



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