# 39-th Austrian Mathematical Olympiad 2008
## Final Round

### Part 1 – May 22

1. What is the remainder of the number
$$1\cdot\binom{2008}{0}+2\cdot\binom{2008}{1}+\cdots+2009\cdot\binom{2008}{2008}$$
when divided by 2008?

2. Given $a\in\mathbb{R}_+$ and an integer $n>4$ determine all $n$-tuples $(x_1,\dots,x_n)$ of positive real numbers that satisfy the following system of equations:
$$\begin{aligned} x_1x_2(3a-2x_3) &= a^3\\ x_2x_3(3a-2x_4) &= a^3\\ &\vdots\\ x_{n-2}x_{n-1}(3a-2x_n) &= a^3\\ x_{n-1}x_n(3a-2x_1) &= a^3\\ x_nx_1(3a-2x_2) &= a^3. \end{aligned}$$

3. Let $p>1$ be a natural number. Consider the set $\mathbb{F}_p$ of all non-consant sequences of non-negative integers that satisfy the recursive relation $a_{n+1}=(p+1)a_n-pa_{n-1}$ for all $n>0$.

   Show that there exists a sequence $(a_n)$ in $\mathbb{F}_p$ with the property that for every other sequence $(b_n)$ in $\mathbb{F}_p$, the inequality $a_n\le b_n$ holds for all $n$.

4. In a triangle $ABC$ let $E$ be the midpoint of the side $AC$ and $F$ the midpoint of the side $BC$. Let $G$ be the foot of the perpendicular from $C$ to $AB$. Show that $\triangle EFG$ is isosceles if and only if $\triangle ABC$ is isosceles.

### Part 2 – June 5–June 6

*First Day*

1. Show that for positive real numbers $a,b,c$ that satisfy $a+b+c=1$ the following inequality holds:
$$\sqrt{a^{1-a}b^{1-b}c^{1-c}}\le\frac{1}{3}.$$

2. (a) Does there exist a polynomial $P(x)$ with integer coefficients such that for each positive integer $d$ that divides 2008 one has $P(d)=\frac{2008}{d}$?



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(b) For which positive integers $n$ do there exist polynomial $P(x)$ with integer coefficients such that for each positive integer $d$ that divides $n$ one has $P(d) = \frac{n}{d}$?

3. Assume that the points $P$, $Q$, $R$, $S$ lie on a line $l$ in this order. Construct all squares $ABCD$ that satisfy: $P \in AD$, $Q \in BC$, $R \in AB$, $S \in CD$ (here $XY$ denotes the line determined by the points $X$ and $Y$).

*Second Day*

4. Determine all functions $f : \mathbb{N} \to \mathbb{N} \cup \{0\}$ that satisfy:

   (i) $f(n \cdot m) = f(n) + f(m)$ for all $n, m \in \mathbb{N}$;

   (ii) $f(2008) = 0$;

   (iii) $f(n) = 0$ whenever $n \equiv 39$ (mod 2008).

5. Determine all positive integers that do not appear in the sequence:
$$\left(n + \left[\sqrt{n}\right] + \left[\sqrt[3]{n}\right]\right)_{n \geq 1}.$$

6. Determine all points $P$ in the plane of the square $ABCD$ different from the vertices and the center of the square for which the line $PD$ intersects the line $AC$ at some point $E$; the line $PC$ intersects the line $DB$ at some point $F$, and $EF \| AD$.