38-th Austrian Mathematical Olympiad 2007
Final Round

Part 1 – May 17

1. In each cell of a 2007×2007 table there is an odd integer. Denote by Z_i the sum of the numbers in the i-th row and by S_j the sum of the numbers in the j-th column. Let $A = \prod_{i=1}^{2007} Z_i$ and $B = \prod_{j=1}^{2007} S_i$. Show that $A + B$ cannot be equal to zero.

2. For each $n \in \mathbb{N}$ find the largest number $C(n)$ such that the inequality

$$(n+1)\sum_{j=1}^{n} a_j^2 - \left(\sum_{j=1}^{n} a_j\right)^2 \geq C(n)$$

   holds for all n-tuples $(a_1, \dots, a_n)$ of pairwise different integers.

3. For each nonempty subset of $M(n) = \{-1, -2, \dots, -n\}$ we compute the product of its elements. What is the sum of all such products?

4. Let $n > 4$ be an integer. An inscribed convex n-gon $A_0 A_1 \dots A_{n-1}$ is given such that its side lengths are $A_{i-1}A_i = i$ for $i = 1, \dots, n$ (where $A_n = A_0$). Denote by ϕ_i the (acute) angle between the line $A_i A_{i+1}$ and the tangent to the circumcircle of the n-gon at A_i. Evaluate the sum $\Phi = \sum_{i=0}^{n-1} \phi_i$.

Part 2 – June 5-6

First Day

1. Find all nonnegative integers $a < 2007$ for which the congruence $x^2 + a \equiv 0$ (mod 2007) has exactly two different nonnegative integer solutions smaller than 2007.

2. Solve in nonnegative integers $x_1, \dots, x_6$ the system of equations

$$x_k x_{k+1}(1 - x_{k+2}) = x_{k+3} x_{k+4}, \quad k = 1, \dots, 6,$$

   where $x_{k+6} = x_k$.

3. Determine all rhombuses with side length $2a$ for which there is a circle cutting a segment of length a from each side of the rhombus.

Second Day

4. Consider the set M of all polynomials $P(x)$ whose all roots are pairwise different integers and whose coefficients are integers less than 2007 in absolute value. What is the highest power among all polynomials in M?

5. A convex n-gon is triangulated, i.e. divided into triangles by nonintersecting diagonals. Prove that the vertices of the n-gon can each be labeled by the digits of number 2007 in such a way that the labels of the vertices of any quadrilateral composed of two adjacent triangles in the triangulation sum up to 9.

6. Let U be the circumcenter of a triangle ABC and P be a point on the extension of UA beyond A. Lines g and h are symmetric to PB and PC with respect to BA and CA, respectively. Let Q be the intersection of g and h. Find the locus of points Q as P takes all possible locations.

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