37-th Austrian Mathematical Olympiad 2008
Final Round

Part 1 – May 21

1. A natural number n ends with exactly k zeros in decimal representation and is greater than 10^k. Find, as a function of k, the smallest possible number of representations of n as a difference of two perfect squares.

2. Prove that the sequence

$$\left\{ \frac{(n+1)^n n^{2-n}}{7n^2+1} \right\}_{n=0,1,\dots}$$

is strictly increasing.

3. The incircle of triangle ABC touches the sides BC and AC at D and E respectively. Prove that if $AD = BE$ then the triangle is isosceles.

4. For each positive number x define $f(x) = [x^2] + \{x\}$ (where $[u]$ is the integral and $\{u\}$ the fractional part of u). Show that there exists a nonconstant arithmetic sequence of positive rational numbers which all have the denominator 3 in the reduced form and none of which occurs as a value of f.

Part 2 – May 31–June 1

First Day

1. Let N be a positive integer. Find the number of natural numbers $n \le N$ which have a multiple whose decimal representation consists of digits 2 and 6 only.

2. If a, b, c are arbitrary positive numbers, prove that

$$3(a+b+c) \ge 8\sqrt[3]{abc} + \sqrt[3]{\frac{a^3+b^3+c^3}{3}}.$$

3. Consider a triangle ABC. The point R on the extension of AB over B is such that $BR = BC$ and the point S on the extension of AC over C is such that $CS = CB$. The diagonals of the quadrilateral $BRSC$ meet at A'. Points B' and C' are similarly constructed. Prove that the area of the hexagon $AC'BA'CB'$ is equal to the sum of the areas of the triangles ABC and $A'B'C'$.

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Second Day

4. For which rational x is $1+105\cdot 2^x$ a square of a rational number?

5. Find all nonincreasing or nondecreasing functions $f:\mathbb{R}\to\mathbb{R}$ that satisfy

$$f(f(x)) = f(-f(x)) = f(x)^2 \quad \text{for all } x.$$

6. Let A be a nonzero integer. Solve the following system in integers:

$$\begin{aligned} x+y^2+z^3 &= A \\ \frac{1}{x}+\frac{1}{y^2}+\frac{1}{z^3} &= \frac{1}{A} \\ xy^2z^3 &= A^2. \end{aligned}$$

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