35-th Austrian Mathematical Olympiad 2004
Final Round

Part 1 – May 16

1. Find all quadruples (a,b,c,d) of real numbers such that
$$a+bcd=b+cda=c+dab=d+abc.$$

2. A convex hexagon $ABCDEF$ with $AB=BC=a$, $CD=DE=b$, $EF=FA=c$ is inscribed in a circle. Show that this hexagon has three (pairwise disjoint) pairs of mutually perpendicular diagonals.

3. For natural numbers a,b, define $Z(a,b)=\dfrac{(3a)!(4b)!}{a!^4b!^3}$.
   (a) Prove that $Z(a,b)$ is an integer for $a \le b$.
   (b) Prove that for each natural number b there are infinitely many natural numbers a such that $Z(a,b)$ is not an integer.

4. Each of the $2N=2004$ real numbers $x_1,x_2,\dots,x_{2004}$ equals either $\sqrt{2}-1$ or $\sqrt{2}+1$. Can the sum $\sum_{k=1}^{N} x_{2k-1}x_{2k}$ take the value 2004? Which integral values can this sum take?

Part 2 – May 26–27

First Day

1. Prove without using advanced (differential) calculus:
   (a) For any real numbers a,b,c,d it holds that $a^6+b^6+c^6+d^6-6abcd \ge -2$. When does equality hold?
   (b) For which natural numbers k does some inequality of the form $a^k+b^k+c^k+d^k-kabcd \ge M_k$ hold for all real a,b,c,d? For each such k, find the greatest possible value of M_k and determine the cases of equality.

2. (a) Given any set $\{p_1,p_2,\dots,p_k\}$ of prime numbers, show that the sum of the reciprocals of all numbers of the form $p_1^{r_1}\cdots p_k^{r_k}$ $(r_1,\dots,r_k\in\mathbb{N})$ is also a reciprocal of an integer.
   (b) Compute the above sum, knowing that $1/2004$ occurs among the summands.
   (c) Prove that for each k-element set $\{p_1,\dots,p_k\}$ of primes ($k>2$), the above sum is smaller than $1/N$, where $N=2\cdot 3^{k-2}(k-2)!$.

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3. A trapezoid $ABCD$ with perpendicular diagonals AC and BD is inscribed in a circle k. Let k_a and k_c respectively be the circles with diameters AB and CD. Compute the area of the region which is inside the circle k, but outside the circles k_a and k_c.

Second Day

4. Show that there is an infinite sequence $a_1, a_2, \dots$ of natural numbers such that $a_1^2 + a_2^2 + \cdots + a_N^2$ is a perfect square for all N. Give a recurrent formula for one such sequence.

5. Solve the following system of equations in real numbers:

$$a^2 = \frac{\sqrt{bc}\sqrt[3]{bcd}}{(b+c)(b+c+d)}, \qquad b^2 = \frac{\sqrt{cd}\sqrt[3]{cda}}{(c+d)(c+d+a)},$$

$$c^2 = \frac{\sqrt{da}\sqrt[3]{dab}}{(d+a)(d+a+b)}, \qquad d^2 = \frac{\sqrt{ab}\sqrt[3]{abc}}{(a+b)(a+b+c)}.$$

6. Outside an equilateral triangle ABC of area 1, three triangles BCP, CAQ, ABR are constructed so that $\angle P = \angle Q = \angle R = 60^\circ$.

   (a) What is the greatest possible area of triangle PQR?

   (b) What is the greatest possible area of the triangle whose vertices are the incenters of triangles BCP, CAQ, ABR?

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