33-rd Austrian Mathematical Olympiad 2002
Final Round

Part 1 – May 28

1. Determine all integers a and b such that
$$(19a+b)^{18}+(a+b)^{18}+(a+19b)^{18}$$
is a perfect square.

2. Find the greatest real number C such that, for all real numbers x and $y \neq x$ with $xy = 2$ it holds that
$$\frac{((x+y)^2-6)((x-y)^2+8)}{(x-y)^2} \geq C.$$
When does equality occur?

3. Let $f(x) = \dfrac{9^x}{9^x+3}$. Compute $\sum_k f\left(\dfrac{k}{2002}\right)$, where k goes over all integers k between 0 and 2002 which are coprime to 2002.

4. Let A, C, P be three distinct points in the plane. Construct all parallelograms $ABCD$ such that point P lies on the bisector of angle DAB and $\angle APD = 90^\circ$.

Part 2 – June 5–6

First Day

1. Consider all possible rectangles that can be drawn on a 8×8 chessboard, covering only whole cells. Calculate the sum of their areas.

   What formula is obtained if "8×8" is replaced with "$a\times b$", where a, b are positive integers?

2. Let b be a natural number. Find all 2002−tuples $(a_1, a_2, \ldots, a_{2002})$, of natural numbers such that
$$\sum_{j=1}^{2002} a_j^{a_j} = 2002b^b.$$

3. Let $ABCD$ and $AEFG$ be two similar cyclic quadrilaterals (with the vertices denoted counterclockwise). Their circumcircles intersect again at point P. Prove that P lies on line BE.

Second Day

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4. Find all polynomials $P(x)$ of the smallest possible degree with the following properties:

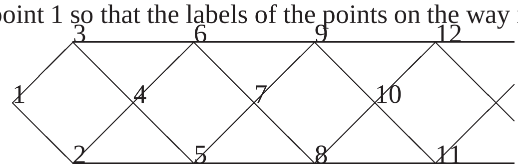
   (i) The leading coefficient is 200;

   (ii) The coefficient at the smallest non-vanishing power is 2;

   (iii) The sum of all the coefficients is 4;

   (iv) $P(-1) = 0$, $P(2) = 6$, $P(3) = 8$.

5. In the net drawn below, in how many ways can one reach the point $3n+1$ starting from the point 1 so that the labels of the points on the way increase?



6. Let H be the orthocenter of an acute-angled triangle ABC. Show that the triangles ABH, BCH and CAH have the same perimeter if and only if the triangle ABC is equilateral.

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