22-nd Austrian–Polish Mathematical Competition 1999

Austria

Individual Competition – June 30 – July 1

First Day

1. Find the number of 6-tuples $(A_1, A_2, \dots, A_6)$ of subsets of $M = \{1, \dots, n\}$ (not necessarily different) such that each element of M belongs to zero, three, or six of the subsets $A_1, \dots, A_6$.

2. Find the largest real number C_1 and the smallest real number C_2 such that for all real numbers a, b, c, d, e the following inequalities hold:

$$C_1 < \frac{a}{a+b} + \frac{b}{b+c} + \frac{c}{c+d} + \frac{d}{d+e} + \frac{e}{e+a} < C_2.$$

3. Given an integer $n \geq 2$, find all sustems of n functions $f_1, \dots, f_n : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$

$$\begin{aligned} f_1(x) - f_2(x) f_2(y) + f_1(y) &= 0 \\ f_2(x^2) - f_3(x) f_3(y) + f_2(y^2) &= 0 \\ &\dots\dots\dots\dots\dots \\ f_n(x^n) - f_1(x) f_1(y) + f_n(y^n) &= 0. \end{aligned}$$

Second Day

4. Three lines k, l, m are drawn through a point P inside a triangle ABC such that k meets AB at A_1 and AC at $A_2 \neq A_1$ and $PA_1 = PA_2$; l meets BC at B_1 and BA at $B_2 \neq B_1$ and $PB_1 = PB_2$; m meets CA at C_1 and CB at $C_2 \neq C_1$ and $PC_1 = PC_2$. Prove that the lines k, l, m are uniquely determined by these conditions. Find point P for which the triangles AA_1A_2, BB_1B_2, CC_1C_2 have the same area and show that this point is unique.

5. A sequence of integers (a_n) satisfies $a_{n+1} = a_n^3 + 1999$ for $n = 1, 2, \dots$. Prove that there exists at most one n for which a_n is a perfect square.

6. Solve in the nonnegative real numbers the system of equations

$$x_n^2 + x_n x_{n-1} + x_{n-1}^4 = 1 \quad \text{for } n = 1, 2, \dots, 1999$$
$$x_0 = x_{1999}.$$

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Team competition – July 2

7. Find all pairs (x, y) of positive integers such that

$$x^{x+y} = y^{y-x}.$$

8. Let P, Q, R be points on the same side of a line g in the plane. Let M and N be the feet of the perpendiculars from P and Q to g respectively. Point S lies between the lines PM and QN and satisfies and satisfies $PM = PS$ and $QN = QS$. The perpendicular bisectors of SM and SN meet in a point R. If the line RS intersects the circumcircle of triangle PQR again at T, prove that S is the midpoint of RT.

9. A point in the cartesian plane with integer coordinates is called a lattice point. Consider the following one player game. A finite set of selected lattice points and finite set of selected segments is called a position in this game if the following hold:

   (i) The endpoints of each selected segment are lattice points;

   (ii) Each selected segment is parallel to a coordinate axis or to one of the lines $y = \pm x$;

   (iii) Each selected segment contains exactly five lattice points, all of which are selected;

   (iv) Every two selected segments have at most one common point.

   A move in this game consists of selecting a lattice point and a segment such that the new set of selected lattice points and segments is a position. Prove or disprove that there exists an initial position such that the game can have infinitely many moves.