21-st Austrian–Polish Mathematical Competition 1998

Przysiek, Poland

Individual Competition – June 29–30

First Day

1. Let x_1, x_2, y_1, y_2 be real numbers such that $x_1^2 + x_2^2 \leq 1$. Prove the inequality
$$(x_1y_1 + x_2y_2 - 1)^2 \geq (x_1^2 + x_2^2 - 1)(y_1^2 + y_2^2 - 1).$$

2. Consider n points $P_1, P_2, \dots, P_n$ lying in this order on a line. Each of these n points is to be colored using one of the following colors: white, red, green, blue, and violet. A coloring is called *admissible* if for any two consecutive points P_i, P_{i+1} $(i = 1, 2, \dots, n-1)$ either both are of the same color or at least one of them is white. How many admissible colorings are there?

3. Find all pairs of real numbers (x, y) satisfying the system of equations
$$2 - x^3 = y, \quad 2 - y^3 = x.$$

Second Day

4. For positive integers m, n, denote
$$S_m(n) = \sum_{1 \leq k \leq n} \left[\sqrt[k^2]{k^m} \right].$$
Prove that $S_m(n) \leq n + m\left(\sqrt[4]{2^m} - 1\right)$.

5. Determine all pairs (a, b) of positive integers for which the equation
$$x^3 - 17x^2 + ax - b^2 = 0$$
has three integer roots (not necessarily different).

6. Different points A, B, C, D, E, F lie on circle k in this order. The tangets to k in the points A and D and the lines BF and CE have a common point P. Prove that the lines AD, BC and EF also have a common point or are parallel.

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Team competition – July 1

7. Consider all pairs (a,b) of natural numbers such that the product $a^a b^b$ written in decimal system ends with exactly 98 zeros. Find the pair (a,b) for which the product ab is the smallest.

8. In each unit square of an infinite square grid a natural number is written. The polygons of area n with sides going along the gridlines are called *admissible*, where $n > 2$ is a given natural number. The *value* of an admissible polygon is defined as the sum of the numbers inside it. Prove that if the values of any two congruent admissible polygons are equal, then all the numbers written in the unit squares of the grid are equal. (We recall that a symmetric image of polygon $\mathscr{P}$ is congruent to $\mathscr{P}$.)

9. Given a triangle ABC, points K,L,M are the midpoints of the sides BC,CA,AB, and points X,Y,Z are the midpoints of the arcs BC,CA,AB of the circumcircle not containing A,B,C respectively. If R denotes the circumradius and r the inradius of the triangle, show that $r + KX + LY + MZ = 2R$.

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