20-th Austrian–Polish Mathematical Competition 1997

Austria

Individual Competition – June 25–26

First Day

1. Lines l_1 and l_2 intersect at point P. Circles S_1 and S_2 are tangent to l_1 at P, and circles T_1 and T_2 are tangent to l_2 at P. Circle S_1 meets T_1 at points A, P and T_2 at B, P, while circle S_2 meets T_2 at C, P and T_1 at D, P. Show that the points A, B, C and D are concyclic if and only if the lines l_1 and l_2 are perpendicular.

2. Each square of an $n \times m$ board is assigned a pair of coordinates (x, y) with $1 \leq x \leq m$ and $1 \leq y \leq n$. Let p and q be positive integers. A pawn can be moved from the square (x, y) to (x', y') if and only if $|x - x'| = p$ and $|y - y'| = q$. There is a pawn on each square. We want to move each pawn at the same time so that no two pawns are moved onto the same square. In how many ways can this be done?

3. The 97 numbers $48/k$, $k = 1, 2, \dots, 97$ are written on the blackboard. In each step two numbers a and b from the blackboard are selected and replaced by $2ab - a - b + 1$. After 96 steps only one number remains. Find all possible values of this number.

Second Day

4. In a trapezoid $ABCD$ with $AB \parallel CD$, the diagonals AC and BD intersect at point E. Let F and G be the orthocenters of the triangles EBC and EAD. Prove that the midpoint of GF lies on the perpendicular from E to AB.

5. Let p_1, p_2, p_3, p_4 be four distinct primes. Prove that there is no polynomial $Q(x) = ax^3 + bx^2 + cx + d$ with integer coefficients such that

$$|Q(p_1)| = |Q(p_2)| = |Q(p_3)| = |Q(p_4)| = 3.$$

6. Prove that there is no function $f : \mathbb{Z} \to \mathbb{Z}$ such that $f(x + f(y)) = f(x) - y$ for all integers x, y.

Team competition – June 27

7. (a) Prove that for any real numbers p, q, $p^2 + q^2 + 1 > p(q + 1)$.

   (b) Determine the largest real constant b such that the inequality $p^2 + q^2 + 1 \geq bp(q + 1)$ holds for all real numbers p, q.

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(c) Determine the largest real constant c such that the inequality $p^2+q^2+1 \geq cp(q+1)$ holds for all integers p,q.

8. Let M be an n-element set. Find the greatest positive integer k with the following property: There exists a k-element family K consisting of 3-element subsets of M, such that every two sets from K have a nonempty intersection.

9. Given a parallelepiped P, let V_P be its volume, S_P the area of its surface and L_P the sum of the lengths of its edges. For a real number $t \geq 0$ let P_t be the solid consisting of all points X whose distance from some point of P is at most t. Prove that the volume of the solid P_t is given by the formula

$$V(P_t) = V_P + S_P t + \frac{\pi}{4} L_P t^2 + \frac{4\pi}{3} t^3.$$

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