19-th Austrian–Polish Mathematical Competition 1996

Zajączkowo, Poland

Individual Competition – June 26–27

First Day

1. Let $k \geq 1$ be a positive integer. Prove that there exist exactly 3^{k-1} natural numbers n with the following properties:

   (i) n has exactly k digits (in decimal representation),

   (ii) all the digits of n are odd,

   (iii) n is divisible by 5,

   (iv) the number $m = n/5$ has k odd digits.

2. A convex hexagon $ABCDEF$ satisfies the following conditions:

   (i) opposite sides are parallel, i.e. $AB \parallel DE$, $BC \parallel EF$, $CD \parallel FA$,

   (ii) the distances between opposite sides are equal,

   (iii) $\angle FAB = \angle CDE = 90^\circ$.

   Prove that the angle between the diagonals BE and CF is equal to 45°.

3. The polynomials $P_n(x)$ are defined recursively by $P_0(x) = 0$, $P_1(x) = x$ and

$$P_n(x) = xP_{n-1}(x) + (1-x)P_{n-2}(x) \quad \text{for } n \geq 2.$$

   For each $n \geq 1$, find all real roots of P_n.

Second Day

4. Real numbers x, y, z, t satisfy $x+y+z+t = 0$ and $x^2+y^2+z^2+t^2 = 1$. Prove that

$$-1 \leq xy + yz + zt + tx \leq 0.$$

5. A sphere $\mathscr{S}$ divides every edge of a convex polyhedron $\mathscr{P}$ into three equal parts. Show that there exists a sphere tangent to all the edges of $\mathscr{P}$.

6. Given natural numbers $n > k > 1$, find all real solutions $x_1, \dots, x_n$ of the system

$$x_i^3(x_i^2 + x_{i+1}^2 + \cdots + x_{i+k-1}^2) = x_{i-1}^2 \quad \text{for } 1 \leq i \leq n.$$

   Here $x_{n+i} = x_i$ for all i.

Team competition – June 28

7. Prove that there are no nonnegative integers k and m such that

$$k! + 48 = 48(k+1)^m.$$

8. Show that there is no polynomial $P(x)$ of degree 998 with real coefficients which satisfies $P(x^2+1) = P(x)^2 - 1$ for all x.

9. For any triple (a,b,c) of positive integers, not all equal, We are given sufficiently many rectangular blocks of size $a \times b \times c$. We use these blocks to fill up a cubic box of edge 10.

   (a) Assume we have used at least 100 blocks. Show that there are two blocks, one of which is a translate of the other.

   (b) Find a number smaller than 100 (the smaller, the better) for which the above statement still holds.

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