14-th Austrian–Polish Mathematical Competition 1991

Bad Ischl, Austria

Individual Competition – June 26-27

First Day

1. Show that there are infinitely many integers $m \geq 2$ such that $\binom{m}{2} = 3\binom{n}{4}$ holds for some integer $n \geq 4$. Give the general form of all such m.

2. Find all solutions (x,y,z) to the system

$$(x^2 - 6x + 13)y = 20$$
$$(y^2 - 6y + 13)z = 20$$
$$(z^2 - 6z + 13)x = 20.$$

3. Given two distinct points A_1, A_2 in the plane, determine all possible positions of a point A_3 with the following property: There exists an array of (not necessarily distinct) points $P_1, P_2, \dots, P_n$ for some $n \geq 3$ such that the segments $P_1P_2, P_2P_3, \dots, P_nP_1$ have equal lengths and their midpoints are $A_1, A_2, A_3, A_1, A_2, A_3, \dots$ in this order.

Second Day

4. Let $P(x)$ be a real polynomial with $P(x) \geq 0$ for $0 \leq x \leq 1$. Show that there exist polynomials $P_i(x)$ $(i = 0,1,2)$ with $P_i(x) \geq 0$ for all real x such that

$$P(x) = P_0(x) + xP_1(x)(1-x)P_2(x).$$

5. If x,y,z are arbitrary positive numbers with $xyz = 1$, prove the inequality

$$x^2 + y^2 + z^2 + xy + yz + zx \geq 2(\sqrt{x} + \sqrt{y} + \sqrt{z}).$$

6. Suppose that there is a point P inside a convex quadrilateral $ABCD$ such that the triangles PAB, PBC, PCD, PDA have equal areas. Prove that one of the diagonals bisects the area of $ABCD$.

Team competition – June 28

7. For a given positive integer n determine the maximum value of the function

$$f(x) = \frac{x + x^2 + \cdots + x^{2n-1}}{(1+x^n)^2}$$

over all $x \geq 0$ and find all positive x for which the maximum is attained.

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com

8. Consider the system of congruences

$$xy \equiv -1 \pmod{z}, \quad yz \equiv 1 \pmod{x}, \quad zx \equiv 1 \pmod{y}.$$

Find the number of triples (x,y,z) of distinct positive integers satisfying this system such that one of the numbers x,y,z equals 19.

9. For a positive integer n denote $A = \{1,2,\dots,n\}$. Suppose that $g : A \to A$ is a fixed function with $g(k) \neq k$ and $g(g(k)) = k$ for $k \in A$. How many functions $f : A \to A$ are there such that

$$f(k) \neq g(k) \quad \text{and} \quad f(f(f(k)) = g(k) \quad \text{for } k \in A?$$