# 12-th Austrian–Polish Mathematical Competition 1989

## Eisenstadt, Austria

### Individual Competition – June 28–29

*First Day*

1. Let $a_k, b_k, c_k$, $k = 1, \dots, n$ be positive numbers. Prove the inequality

$$\left(\sum_{k=1}^{n} a_k b_k c_k\right)^3 \le \left(\sum_{k=1}^{n} a_k^3\right)\left(\sum_{k=1}^{n} b_k^3\right)\left(\sum_{k=1}^{n} c_k^3\right).$$

2. Each point of the plane is colored by one of the two colors. Show that there exists an equilateral triangle with monochromatic vertices.

3. Find all natural numbers $N$ (in decimal system) with the following properties:

   (i) $N = \overline{aabb}$, where $\overline{aab}$ and $\overline{abb}$ are primes;

   (ii) $N = P_1P_2P_3$, where $P_k$ $(k = 1,2,3)$ is a prime consisting of $k$ (decimal) digits.

*Second Day*

4. Let $\mathscr{P}$ be a convex polygon in the plane. Show that there exists a circle containing the entire polygon $\mathscr{P}$ and having at least three adjacent vertices of $\mathscr{P}$ on its boundary.

5. Let $A$ be a vertex of a cube $\omega$ circumscribed about a sphere $\kappa$ of radius 1. We consider lines $g$ through $A$ containing at least one point of $\kappa$. Let $P$ be the intersection point of $g$ and $\kappa$ closer to $A$, and $Q$ be the second intersection point of $g$ and $\omega$. Determine the maximum value of $AP \cdot AQ$ and characterize the lines $g$ yielding the maximum.

6. A sequence $(a_n)_{n\in\mathbb{N}}$ of squares of nonzero integers is such that for each $n$ the difference $a_{n+1} - a_n$ is a prime or the square of a prime. Show that all such sequences are finite and determine the longest sequence.

### Team competition – June 30

7. Functions $f_0, f_1, f_2, \dots$ are recursively defined by $f_0(x) = x$ and

$$f_{2k+1}(x) = 3^{f_{2k}(x)} \quad \text{and} \quad f_{2k+2} = 2^{f_{2k+1}(x)}, \ k = 0,1,2,\dots$$

for all $x \in \mathbb{R}$. Find the greater one of the numbers $f_{10}(1)$ and $f_9(2)$.



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8. An acute triangle $ABC$ is given. For each point $P$ of the interior or boundary of $\triangle ABC$, $P_a, P_b, P_c$ denote the orthogonal projections of $P$ to $BC, CA, AB$ respectively. Consider
$$f(P) = \frac{AP_c + BP_a + CP_b}{PP_a + PP_b + PP_c}.$$
Show that $f(P)$ is constant if and only if $ABC$ is an equilateral triangle.

9. Find the smallest odd natural number $N$ such that $N^2$ is the sum of an odd number (greater than 1) of squares of adjacent positive integers.



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