11-th Austrian–Polish Mathematical Competition 1988

Koszalin, Poland

Individual Competition – July 6–7

First Day

1. Let $P(x)$ be a polynomial with integer coefficients. Show that if $Q(x)=P(x)+12$ has at least six distinct integer roots, then $P(x)$ has no integer roots.

2. If $a_1 \le a_2 \le \cdots \le a_n$ are natural numbers $(n \ge 2)$, show that the inequality

$$\sum_{i=1}^{n} a_i x_i^2 + 2\sum_{i=1}^{n-1} x_i x_{i+1} > 0$$

   holds for all n-tuples $(x_1,\dots,x_n) \ne (0,\dots,0)$ of real numbers if and only if $a_2 \ge 2$.

3. Let $ABCD$ be a convex quadrilateral with no two parallel sides. Consider the two angles made by two pairs of opposite sides. Their angle bisectors intersect the sides of $ABCD$ in P,Q,R,S, where $PQRS$ is a convex quadrilateral. Prove that the quadrilateral $ABCD$ is cyclic if and only if $PQRS$ is a rhombus.

Second Day

4. Determine all strictly increasing functions $f:\mathbb{R}\to\mathbb{R}$ satisfying

$$f(f(x)+y) = f(x+y)+f(0) \quad \text{for all } x,y \in \mathbb{R}.$$

5. Two sequences $(a_k)_{k\ge 0}$ and $(b_k)_{k\ge 0}$ of integers are given by

$$b_k = a_k + 9 \ \text{ and } \ a_{k+1} = 8b_k + 8 \quad \text{for } k \ge 0.$$

   Suppose that the number 1988 occurs in one of these sequences. Show that the sequence (a_k) does not contain any nonzero perfect square.

6. Three rays h_1,h_2,h_3 emanating from a point O are given, not all in the same plane. Show that if for any three points A_1,A_2,A_3 on h_1,h_2,h_3 respectively, distinct from O, the triangle $A_1A_2A_3$ is acute-angled, then the rays h_1,h_2,h_3 are pairwise orthogonal.

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Team competition – July 8

7. Each side of a regular octagon is colored blue or yellow. In each step, the sides are simultaneously recolored as follows: if the two neighbors of a side have different colors, the side will be recolored blue, otherwise it will be recolored yellow. Show that after a finite number of moves all sides will be colored yellow. What is the least value of the number N of moves that always lead to all sides being yellow?

8. We are given 1988 unit cubes. Using some or all of these cubes, we form three quadratic boards A, B, C of dimensions $a \times a \times 1$, $b \times b \times 1$, and $c \times c \times 1$ respectively, where $a \leq b \leq c$. Now we place board B on board C so that each cube of B is precisely above a cube of C and B does not overlap C. Similarly, we place A on B. This gives us a three-floor tower. What choice of a, b and c gives the maximum number of such three-floor towers?

9. For a rectangle $\mathscr{R}$ with integral side lengths, denote by $D(a,b)$ the number of ways of covering $\mathscr{R}$ by congruent rectangles with integral side lengths formed by a family of cuts parallel to one side of $\mathscr{R}$. Determine the perimeter P of the rectangle $\mathscr{R}$ for which $\frac{D(a,b)}{a+b}$ is maximal.

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