# 6-th Austrian–Polish Mathematical Competition 1983

???, ???

**Individual Competition** – June ??-??

*First Day*

1. Nonnegative real numbers $a,b,x,y$ satisfy $a^5+b^5 \leq 1$ and $x^5+y^5 \leq 1$. Prove that $a^2x^3+b^2y^3 \leq 1$.

2. Find all triples of positive integers $(p,q,n)$ with $p$ and $q$ prime, such that
$$p(p+1)+q(q+1)=n(n+1).$$

3. A bounded planar region of area $S$ is covered by a finite family $\mathscr{F}$ of closed discs. Prove that $\mathscr{F}$ contains a subfamily consisting of pairwise disjoint discs, of joint area not less than $S/9$.

*Second Day*

4. The set $\mathbb{N}$ has been partitioned into two sets $A$ and $B$. Show that for every $n \in \mathbb{N}$ there exist distinct integers $a,b > n$ such that $a,b,a+b$ either all belong to $A$ or all belong to $B$.

5. Let $a_1 < a_2 < a_3 < a_4$ be given positive numbers. Find all real values of parameter $c$ for which the system
$$\begin{array}{rcrcrcrcl} x_1 & + & x_2 & + & x_3 & + & x_4 & = & 1 \\ a_1x_1 & + & a_2x_2 & + & a_3x_3 & + & a_4x_4 & = & c \\ a_1^2x_1 & + & a_2^2x_2 & + & a_3^2x_3 & + & a_4^2x_4 & = & c^2 \end{array}$$
has a solution in nonnegative $(x_1,x_2,x_3,x_4)$ real numbers.

6. Six straight lines are given in space. Among any three of them, two are perpendicular. Show that the given lines can be labeled $l_1,\dots,l_6$ in such a way that $l_1,l_2,l_3$ are pairwise perpendicular, and so are $l_4,l_5,l_6$.

**Team competition** – June ??

7. Let $P_1,P_2,P_3,P_4$ be four distinct points in the plane. Suppose $I_1,I_2,\dots,I_6$ are closed segments in that plane with the following property: Every straight line passing through at least one of the points $P_i$ meets the union $I_1 \cup I_2 \cup \cdots \cup I_6$ in exactly two points. Prove or disprove that the segments $I_j$ necessarily form a hexagon.



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8. (a) Prove that $(2^{n+1}-1)!$ is divisible by $\prod_{i=0}^{n}(2^{n+1-i}-1)^{2^i}$, for every natural number $n$.

   (b) Define the sequence $(c_n)$ by $c_1 = 1$ and $c_n = \frac{4n-6}{n}c_{n-1}$ for $n \geq 2$. Show that each $c_n$ is an integer.

9. To each side of the regular $p$-gon of side length 1 there is attached a $1 \times k$ rectangle, partitioned into $k$ unit cells, where $k$ and $p$ are given positive integers and $p$ an odd prime. Let $\mathscr{P}$ be the resulting nonconvex star-like polygonal figure consisting of $kp+1$ regions ($kp$ unit cells and the $p$-gon). Each region is to be colored in one of three colors, adjacent regions having different colors. Furthermore, it is required that the colored figure should not have a symmetry axis. In how many ways can this be done?



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