5-th Austrian–Polish Mathematical Competition 1982

Toruń, Poland

Individual Competition – June 9–10

First Day

1. Find all pairs (n,m) of positive integers such that
$$\gcd\left((n+1)^m - n, (n+1)^{m+3} - n\right) > 1.$$

2. Let $\mathscr{F}$ be a closed convex region inside a circle $\mathscr{C}$ with center O and radius 1. Furthermore, assume that from each point of $\mathscr{C}$ one can draw two rays tangent to $\mathscr{F}$ which form an angle of 60°. Prove that $\mathscr{F}$ is the disc centered at O with radius $1/2$.

3. If $n \geq 2$ is an integer, prove the equality
$$\prod_{k=1}^{n} \tan \frac{\pi}{3}\left(1 + \frac{3^k}{3^n - 1}\right) = \prod_{k=1}^{n} \cot \frac{\pi}{3}\left(1 - \frac{3^k}{3^n - 1}\right).$$

Second Day

4. Let $P(x)$ denote the product of all (decimal) digits of a natural number x. For any positive integer x_1, define the sequence (x_n) recursively by $x_{n+1} = x_n + P(x_n)$. Prove or disprove that the sequence (x_n) is necessarily bounded.

5. Suppose that the closed interval $[0,1]$ has been partitioned into two (disjoint) subsets A and B. Show that there is no real number a such that $B = A + a$ (where $A + a = \{x + a \mid x \in A\}$).

6. An integer a is given. Find all real-valued functions $f(x)$ defined on integers $x \geq a$, satisfying the equation
$$f(x+y) = f(x)f(y) \quad \text{for all } x,y \geq a \text{ with } x+y \geq a.$$

Team competition – June 12

7. Find the triple of positive integers (x,y,z) with z least possible for which there are positive integers a,b,c,d with the following properties:

   (i) $x^y = a^b = c^d$ and $x > a > c$;

   (ii) $z = ab = cd$;

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(iii) $x+y=a+b$.

8. Let P be a point inside a regular tetrahedron $ABCD$ with edge length 1. Show that

$$d(P,AB)+d(P,AC)+d(P,AD)+d(P,BC)+d(P,BD)+d(P,CD)\geq \frac{3}{2}\sqrt{2},$$

with equality only when P is the centroid of $ABCD$. Here $d(P,XY)$ denotes the distance from point P to line XY.

9. Define $S_n=\sum_{j,k=1}^{n}\frac{1}{\sqrt{j^2+k^2}}$. Find a positive constant C such that the inequality $n\leq S_n\leq Cn$ holds for all $n\geq 3$. (*Note.* The smaller C, the better the solution.)