3-rd Austrian–Polish Mathematical Competition 1980

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Individual Competition – July 3–4

First Day

1. Three infinite arithmetic progressions with positive integer terms are given. Assuming that each of the numbers $1,2,3,4,5,6,7,8$ occurs in at least one of these progressions, show that 1980 must occur in at least one of them.

2. A sequence of integers $1 = x_1 < x_2 < x_3 < \cdots$ satisfies $x_{n+1} \le 2n$ for all n. Show that every positive integer k can be written as $x_j - x_i$ for some i, j.

3. Prove that for every point P inside a regular tetrahedron $ABCD$ the sum of the angles $APB, APC, APD, BPC, BPD, CPD$ exceeds 540°.

Second Day

4. Prove that
$$\sum \frac{1}{i_1 i_2 \cdots i_k} = n,$$
summation going over all nonempty subsets $\{i_1, \dots, i_k\}$ of $\{1, \dots, n\}$.

5. Let B_1, B_2, B_3 be points on sides A_2A_3, A_3A_1, A_1A_2 respectively of a triangle $A_1A_2A_3$ (not coinciding with any vertices). Prove that the perpendicular bisectors of the three segments A_iB_i never concur.

6. The sequence $a_1, a_2, a_3, \dots$ has the property that $|a_{k+m} - a_k - a_m| \le 1$ for all k and m. Show that for every $k, m \in \mathbb{N}$,
$$\left|\frac{a_k}{k} - \frac{a_m}{m}\right| < \frac{1}{k} + \frac{1}{m}.$$

Team competition – July 5

7. Find the greatest $n \in \mathbb{N}$ for which there exist positive integers $x_1, x_2, \dots, x_n$ and $a_1, a_2, \dots, a_{n-1}$ with $a_1 < \cdots < a_{n-1}$ such that $x_1 x_2 \cdots x_n = 1980$ and $x_i + \dfrac{1980}{x_i} = a_i$ for all $i = 1, 2, \dots, n-1$.

8. Let S be a set of 1980 points in the plane such that every two points of S are at least 1 apart. Prove that S contains a subset T of 220 points, every two at least $\sqrt{3}$ apart.

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9. Through the endpoints A and B of a diameter AB of a given circle, the tangents l and m have been drawn. Let $C \neq A$ be a point on l and let q_1, q_2 be two rays from C. Ray q_i cuts the circle in D_i and E_i with D_i between C and E_i, $i = 1, 2$. Rays AD_1, AD_2, AE_1, AE_2 meet m in the respective points M_1, M_2, N_1, N_2. Prove that $M_1M_2 = N_1N_2$.

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