2-nd Austrian–Polish Mathematical Competition 1979

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Individual Competition – June ??

First Day

1. On sides AB and BC of a square $ABCD$ the respective points E and F have been chosen so that $BE = BF$. Let BN be the altitude in triangle BCE. Prove that $\angle DNF = 90^\circ$.

2. Find all polynomials of the form
$$P_n(x) = n!x^n + a_{n-1}x^{n-1} + \cdots + a_1x + (-1)^n n(n+1)$$
with integer coefficients, having n real roots $x_1, \dots, x_n$ satisfying $k \le x_k \le k+1$ for $k = 1, \dots, n$.

3. Find all positive integers n such that the inequality
$$\left(\sum_{i=1}^n a_i^2\right)\left(\sum_{i=1}^n a_i\right) - \sum_{i=1}^n a_i^3 \ge 6\prod_{i=1}^n a_i$$
holds for any n positive numbers $a_1, \dots, a_n$.

Second Day

4. Determine all functions $f : \mathbb{N}_0 \to \mathbb{R}$ satisfying
$$f(x+y) + f(x-y) = f(3x) \quad \text{for all } x, y.$$

5. The circumcenter and incenter of a given tetrahedron coincide. Prove that all its faces are congruent.

6. A positive integer n and a real number a are given. Find all n-tuples $(x_1, \dots, x_n)$ of real numbers that satisfy the system of equations
$$\sum_{i=1}^n x_i^k = a^k \quad \text{for } k = 1, 2, \dots, n.$$

Team competition – June ??

7. Let n and m be fixed positive integers. The hexagon $ABCDEF$ with vertices $A = (0,0)$, $B = (n,0)$, $C = (n,m)$, $D = (n-1,m)$, $E = (n-1,1)$, $F = (0,1)$ has been partitioned into $n+m-1$ unit squares. Find the number of paths from A to C along grid lines, passing through every grid node at most once.

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8. Let A, B, C, D be four points in space, and M and N be the midpoints of AC and BD, respectively. Show that

$$AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2 + 4MN^2.$$

9. Find the greatest power of 2 that divides $a_n = \left[(3+\sqrt{11})^{2n+1}\right]$, where n is a given positive integer.

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