1-st Austrian–Polish Mathematical Competition 1978

???, ??

Individual Competition – June ??–??

First Day

1. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}$ which satisfy
$$f(x+y) = f(x^2+y^2) \quad \text{for all } x, y > 0.$$

2. A parallelogram is inscribed into a regular hexagon so that the centers of symmetry of both figures coincide. Prove that the area of the parallelogram is at most $2/3$ the area of the hexagon.

3. Show that
$$\sqrt[44]{\tan 1^\circ \cdot \tan 2^\circ \cdots \tan 44^\circ} < \sqrt{2} - 1 < \frac{\tan 1^\circ + \cdots + \tan 44^\circ}{44}.$$

Second Day

4. Given a positive rational number $c \neq 1$, show that one can partition $\mathbb{N}$ into two nonempty subsets A and B such that $\frac{x}{y} \neq c$ whenever x, y are both in A or both in B.

5. We are given 1978 sets, each containing 40 elements. Every two sets have exactly one element in common. Prove that all 1978 sets have a common element.

6. A family of disks with pairwise disjoint interiors is given in the plane. Each disk is tangent to at least six other disks in the family. Show that the family is infinite.

Team competition – June ??

7. Let M be the set of integer points in the coordinate plane. For any point $(x,y) \in M$ we call the points $(x \pm 1, y), (x, y \pm 1)$ *neighbors* of P. Let S be a given finite subset of M. A one-to-one mapping $f : S \to S$ is called *perfect* if $f(P)$ is a neighbor of P for any $P \in S$. Prove that if such a mapping exists, then there exists also a perfect mapping $g : S \to S$ satisfying $g(g(P)) = P$ for any $P \in S$.

8. For any positive integer k, define
$$a_n = \sqrt{k + \sqrt{k + \cdots + \sqrt{k}}} \quad (n \text{ square roots.}$$

(a) Show that the sequence converges for every fixed k.

(b) Find all k for which this limit is an integer. Furthermore, prove that if k is odd then the limit is irrational.

9. In a convex polygon $\mathscr{P}$ some diagonals have been drawn, without intersections inside $\mathscr{P}$. Show that there are at least two vertices of $\mathscr{P}$, neither of which being an endpoint of any of the drawn diagonals.

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