25-th Austrian–Polish Mathematical Competition 2002

Pułtusk, Poland, June 2002

Individual Competition

First Day

1. Find all triples (a,b,c) of nonnegative integers such that 2^c-1 divides 2^a+2^b+1.

2. Prove that in any convex polygon $P_1P_2\dots P_{2n}$ with an even number of vertices there exists a diagonal P_iP_j which is not parallel to any of its sides.

3. Let S be the centroid of a tetrahedron $ABCD$. A line through S intersects the surface of the tetrahedron at points K and L. Prove that $\frac{1}{3} \leq \frac{KS}{LS} \leq 3$.

Second Day

4. For each positive integer n find a maximum subset $M(n)$ of the set of real numbers such that any elements $x_1,\dots,x_n \in M(n)$ satisfy

$$n+\sum_{i=1}^{n} x_i^{n+1} \geq n\prod_{i=1}^{n} x_i + \sum_{i=1}^{n} x_i.$$

When does equality occur?

5. Consider the set $A=\{2,7,11,13\}$. A polynomial f with integer coefficients has the property that for each integer n, $f(n)$ is divisible by some prime from A. Prove that there exists $p \in A$ such that $p \mid f(n)$ for all integers n.

6. The diagonals of a convex quadrilateral $ABCD$ meet at E. Let U and H be the circumcenter and orthocenter of triangle ABE, respectively. Similarly, let V and K be the circumcenter and orthocenter of triangle CDE, respectively. Prove that E lies on line UK if and only if it lies on line VH.

Team competition

7. Find all functions $f:\mathbb{N}\to\mathbb{R}$ satisfying $f(x+22)=f(x)$ and $f(x^2y)=f(x)^2f(y)$ for all positive integers x and y.

8. For each $n \in \mathbb{N}$, determine the number of real solutions of the system

$$\cos x_1 = x_2, \quad \cos x_2 = x_3, \quad \dots \quad \cos x_n = x_1.$$

9. A set P of 2002 persons is given. Suppose that the number of acquaintance pairs in every 1001-element subset of P is the same (the acquaintance relation is symmetric). Find the best lower bound for the number of acquaintance pairs in P.

10. For each real number x consider the family F_x of all sequences $(a_n)_{n\geq 0}$ satisfying the relation $a_{n+1} = x - \frac{1}{a_n}$ for all n.
A positive integer p is called the *minimum period* of F_x if (i) each sequence in F_x has a period p and (ii) for any $0 < q < p$ there is a sequence in F_x which is not periodic with period q.
Prove or disprove that for each positive integer P there exists a real number x such that the family F_x has a minimum period $p > P$.

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