24-th Austrian–Polish Mathematical Competition 2001

St. Georgen im Attergau, Austria

Individual Competition

First Day

1. Determine the number of positive integers a for which there exist nonnegative integers $x_0, x_1, \dots, x_{2001}$ such that $a^{x_0} = a^{x_1} + \cdots + a_{x_{2001}}$.

2. Given an integer $n > 2$, solve in nonnegative real numbers the system

$$x_k + x_{k+1} = x_{k+2}^2, \quad k = 1, 2, \dots, n,$$

where $x_{n+i} = x_i$.

3. If a, b, c are side lengths of a triangle, prove the inequality

$$2 < \frac{a+b}{c} + \frac{b+c}{a} + \frac{c+a}{b} - \frac{a^3+b^3+c^3}{abc} \le 3.$$

Second Day

4. Prove that if a, b, c, d are lengths of the successive sides of a quadrilateral (not necessarily convex) and S its area, then $S \le \frac{1}{2}(ac+bd)$. When does equality hold?

5. The fields of an 8×8 chessboard are numbered from 1 to 64 in such a way that for each $i = 1, 2, \dots, 63$ the field $i+1$ can be reached from the field i by a move of a knight. Let $x_1, x_2, \dots, x_{64}$ be positive numbers. Define

$$y_i = 1 + x_i^2 - \sqrt[3]{x_{i-1}^2 x_{i+1}} \quad \text{if field } i \text{ is white and}$$
$$y_i = 1 + x_i^2 - \sqrt[3]{x_{i-1} x_{i+1}^2} \quad \text{if field } i \text{ is black,}$$

where $x_{64+i} = x_i$. Prove that $\sum_{i=1}^{64} y_i \ge 48$.

6. Let k be a fixed positive integer. Consider the sequence defined by $a_0 = 1$ and

$$a_{n+1} = a_n + [\sqrt[k]{a_n}], \quad n = 0, 1, \dots$$

For each k find the set A_k of all integer values of the sequence $\sqrt[k]{a_n}$, $n \ge 0$.

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Team competition

7. Consider the set A of all positive integer containing no zero (decimal) digit and which are divisible by their sum of digits.

   (a) Prove that A contains infinitely many numbers whose decimal expansion contains each of its digits the same number of times.

   (b) Show that for each $k \in \mathbb{N}$ there is a k-digit number in A.

8. A prism with the regular octagonal base and all edges of the length 1 is given. Let $M_1, M_2, \dots, M_{10}$ be the centers of the faces of the prism. For a point P inside the prism denote by P_i the second intersection point of line PM_i with the surface of the prism. Assume that the interior of each face contains exactly one of the points P_i. Prove that

$$\sum_{i=1}^{10} \frac{M_iP}{M_iP_i} = 5.$$

9. Consider a $2n$-element set A, where $n > 10$ is an integer. A family of subsets $\{A_i \mid i = 1, 2, \dots, m\}$ is called *suitable* if

   (i) for each i the set A_i contains exactly n elements, and

   (ii) for all distinct i, j, k the set $A_i \cap A_j \cap A_k$ contains at most one element.

   For each n determine the maximum size of a suitable family.

10. The sequence $a_1, a_2, \dots, a_{2010}$ has the following properties:

    (i) The sum of any 20 consecutive terms is nonnegative;

    (ii) $|a_i a_{i+1}| \le 1$ for $i = 1, 2, \dots, 2009$.

    Determine the maximum possible value of the sum $a_1 + a_2 + \cdots + a_{2010}$.

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