23-rd Austrian–Polish Mathematical Competition 2000

Baranów Sandomierski, Poland

Individual Competition – June 28–29

First Day

1. Find all polynomials $P(x)$ with real coefficients having the following property: There exists a positive integer n such that the equality

$$\sum_{k=1}^{2n+1}(-1)^k\left[\frac{k}{2}\right]P(x+k)=0$$

holds for infinitely many real numbers x.

2. In a unit cube, CG is the edge perpendicular to the face $ABCD$. Let O_1 be the incircle of square $ABCD$ and O_2 be the circumcircle of triangle BDG. Determine $\min\{XY \mid X\in O_1, Y\in O_2\}$.

3. For each integer $n\geq 3$ solve in real numbers the system of equations:

$$\begin{aligned} x_1^3 &= x_2+x_3+1 \\ &\dots \\ x_{n-1}^3 &= x_n+x_1+1 \\ x_n^3 &= x_1+x_2+1. \end{aligned}$$

Second Day

4. Find all positive integers N possessing only 2 and 5 as prime divisors, such that $N+25$ is a square.

5. For which integers $n\geq 5$ is it possible to color the vertices of a regular n-gon using at most 6 colors in such a way that any 5 consecutive vertices have different colors?

6. Consider the solid Q obtained by attaching unit cubes $Q_1,\dots,Q_6$ at the six faces of a unit cube Q. Prove or disprove that the space can be filled up with such solids so that no two of them have a common interior point.

Team competition – June 30

7. Triangle $A_0B_0C_0$ is given in the plane. Consider all triangles ABC such that:

   (i) The lines AB, BC, CA pass through C_0, A_0, B_0, respectvely;

(ii) The triangles ABC and $A_0B_0C_0$ are similar.

Find the possible positions of the circumcenter of triangle ABC.

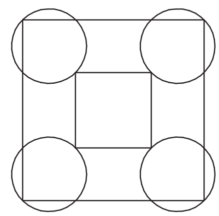
8. In the plane are given 27 points, no three of which are collinear. Four of this points are vertices of a unit square, while the others lie inside the square. Prove that there are three points in this set forming a triangle with area not exceeding $1/48$.

9. If a, b, c are nonnegative real numbers with $a+b+c=1$, prove that

$$2 \leq (1-a^2)^2 + (1-b^2)^2 + (1-c^2)^2 \leq (1+a)(1+b)(1+c).$$

For both inequalities determine the cases of equality.

10. The plan of the castle in Baranów Sandomierski can be presented as the graph with 16 vertices on the picture. A night guard plans a closed round along the edges of this graph.



(a) How many rounds passing through each vertex exactly once are there? The directions are irrelevant.

(b) How many non-selfintersecting rounds (taking directions into account) containing each edge of the graph exactly once are there?

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