8-th Asian–Pacific Mathematical Olympiad 1996

1. Let $ABCD$ be a rhombus and let MN and PQ be two segments perpendicular to the diagonal BD such that the distance between them is $d > BD/2$, with $M \in AD$, $N \in DC$, $P \in AB$, and $Q \in BC$. Show that the perimeter of hexagon $AMNCQP$ does not depend on the position of MN and PQ so long as the distance between them remains constant.

2. Let m and n be positive integers with $n \leq m$. Prove that

$$2^n n! \leq \frac{(m+n)!}{(m-n)!} \leq (m^2+m)^n.$$

3. Let P_1, P_2, P_3, P_4 be four points on a circle, and let I_1, I_2, I_3, I_4 respectively be the incenters of the triangles $P_2P_3P_4, P_1P_3P_4, P_1P_2P_4$, and $P_1P_2P_3$. Prove that I_1, I_2, I_3, I_4 are the vertices of a rectangle.

4. The National Marriage Council wishes to invite n coples to form 17 discussion groups under the following conditions:

   (i) All members of a group must be of the same sex.

   (ii) The difference in the size of any two groups is 0 or 1.

   (iii) Each group has at least one member.

   (iv) Each person must belong to one and only one group.

   Find all values of $n \leq 1996$ for which this is possible. Justify your answer.

5. Let a, b, c be the lengths of the sides of a triangle. Prove that

$$\sqrt{a+b-c} + \sqrt{b+c-a} + \sqrt{c+a-b} \leq \sqrt{a} + \sqrt{b} + \sqrt{c},$$

   and determine when equality occurs.

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