16-th Asian–Pacific Mathematical Olympiad 2004

1. Determine all finite nonempty sets S of positive integers with the following property: For any $i, j \in S$, number $\frac{i+j}{(i,j)}$ is an element of S.

2. Let O be the circumcenter and H the orthocenter of an acute triangle ABC. Prove that the area of one of the triangles AOH, BOH and COH is equal to the sum of the areas of the other two.

3. Let a set S of 2004 points in the plane be given, no three of which are collinear. Let $\mathscr{L}$ denote the set of all lines determined by pairs of points from the set. Show that it is possible to color the points of S with at most two colors, such that for any two points p, q of S, the number of lines in $\mathscr{L}$ which separate p from q is odd if and only if p and q have the same color.

4. Prove that $\left\lfloor \frac{(n-1)!}{n(n+1)} \right\rfloor$ is even for every positive integer n.

5. Prove that for all real numbers $a, b, c > 0$,
$$(a^2+2)(b^2+2)(c^2+2) \geq 9(ab+bc+ca).$$

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com