15-th Asian–Pacific Mathematical Olympiad 2003

1. Let a, b, c, d, e, f be real numbers such that the polynomial

$$p(x) = x^8 - 4x^7 + 7x^6 + ax^5 + bx^4 + cx^3 + dx^2 + ex + f$$

factorises into eight linear factors $x - x_i$, with $x_i > 0$ for $i = 1, 2, \dots, 8$. Determine all possible values of f.

2. Suppose $ABCD$ is a square piece of cardboard with side length a. On a plane are two parallel lines ℓ_1 and ℓ_2, which are also a units apart. The square $ABCD$ is placed on the plane so that sides AB and AD intersect ℓ_1 at E and F respectively. Also, sides CB and CD intersect ℓ_2 at G and H respectively. Let the perimeters of $\triangle AEF$ and $\triangle CGH$ be m_1 and m_2 respectively. Prove that no matter how the square was placed, $m_1 + m_2$ remains constant.

3. Let $k \geq 14$ be an integer, and let p_k be the largest prime number which is strictly less than k. You may assume that $p_k \geq 3k/4$. Let n be a composite integer. Prove:

   (a) if $n = 2p_k$, then n does not divide $(n-k)!$;

   (b) if $n > 2p_k$, then n divides $(n-k)!$.

4. Let a, b, c be the sides of a triangle, with $a + b + c = 1$, and let $n \geq 2$ be an integer. Show that

$$\sqrt[n]{a^n + b^n} + \sqrt[n]{b^n + c^n} + \sqrt[n]{c^n + a^n} < 1 + \frac{\sqrt[n]{2}}{2}\,.$$

5. Given two positive integers m and n, find the smallest positive integer k such that among any k people, either there are $2m$ of them who form m pairs of mutually acquainted people or there are $2n$ of them forming n pairs of mutually unacquainted people.

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