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IMO Shortlist 2009

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1

Problems

1.1 The Fiftieth IMO
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1.1.1 Contest Problems

First Day (July 15)

1. Let n be a positive integer and let $a_1, \ldots, a_k$ ($k \geq 2$) be distinct integers in the set $\{1, \ldots, n\}$ such that n divides $a_i(a_{i+1} - 1)$ for $i = 1, \ldots, k-1$. Prove that n does not divide $a_k(a_1 - 1)$.

2. Let ABC be a triangle with circumcenter O. The points P and Q are interior points of the sides CA and AB, respectively. Let K, L and M be the midpoints of the segments BP, CQ, and PQ, respectively, and let Γ be the circle passing through K, L, and M. Suppose that the line PQ is tangent to the circle Γ. Prove that $OP = OQ$.

3. Suppose that $s_1, s_2, s_3, \ldots$ is a strictly increasing sequence of positive integers such that the subsequences

$$s_{s_1}, s_{s_2}, s_{s_3}, \ldots \quad \text{and} \quad s_{s_1+1}, s_{s_2+1}, s_{s_3+1}, \ldots$$

are both arithmetic progressions. Prove that the sequence $s_1, s_2, s_3, \ldots$ is itself an arithmetic progression.

Second Day (July 16)

4. Let ABC be a triangle with $AB = AC$. The angle bisectors of $\angle CAB$ and $\angle ABC$ meet the sides BC and CA at D and E, respectively. Let K be the incenter of triangle ADC . Suppose that $\angle BEK = 45^\circ$. Find all possible values of $\angle CAB$.

5. Determine all functions f from the set of positive integers to the set of positive integers such that, for all positive integers a and b, there exists a non-degenerate triangle with sides of lengths

$$a,\ f(b)\ \text{ and }\ f(b+f(a)-1).$$

(A triangle is *non-degenerate* if its vertices are not collinear.)

6. Let $a_1, a_2, ..., a_n$ be distinct positive integers and let M be a set of $n-1$ positive integers not containing $s = a_1 + a_2 + \cdots + a_n$. A grasshopper is to jump along the real axis, starting at the point 0 and making n jumps to the right with lengths $a_1, a_2, \ldots, a_n$ in some order. Prove that the order can be chosen in such a way that the grasshopper never lands on any point in M.

1.1.2 Shortlisted Problems

1. **A1 (CZE)** Find the largest possible integer k such that the following statement is true:
   Let 2009 arbitrary non-degenerated triangles be given. In every triangle the three sides are colored, such that one is blue, one is red, and one is white. Now, for every color separately, let us sort the lengths of the sides. We obtain

$$\begin{array}{rll} & b_1 \le b_2 \le \cdots \le b_{2009} & \text{the lengths of the blue sides,} \\ & r_1 \le r_2 \le \cdots \le r_{2009} & \text{the lengths of the red sides,} \\ \text{and} & w_1 \le w_2 \le \cdots \le w_{2009} & \text{the lengths of the white sides.} \end{array}$$

   Then there exist k indeces j such that we can form a non-degenerated triangle with side lengths b_j, r_j, w_j.

2. **A2 (EST)** Let a, b, c be positive real numbers such that $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=a+b+c$. prove that

$$\frac{1}{(2a+b+c)^2}+\frac{1}{(2b+c+a)^2}+\frac{1}{(2c+a+b)^2}\le\frac{3}{16}.$$

3. **A3 (FRA)** IMO5 Determine all functions f from the set of positive integers to the set of positive integers such that, for all positive integers a and b, there exists a non-degenerate triangle with sides of lengths

$$a,\ f(b)\ \text{ and }\ f(b+f(a)-1).$$

   (A triangle is *non-degenerate* if its vertices are not collinear.)

4. **A4 (BEL)** Let a, b, c be positive real numbers such that $ab+bc+ca \le 3abc$. Prove that

$$\sqrt{\frac{a^2+b^2}{a+b}}+\sqrt{\frac{b^2+c^2}{b+c}}+\sqrt{\frac{c^2+a^2}{c+a}}+3\le\sqrt{2}\left(\sqrt{a+b}+\sqrt{b+c}+\sqrt{c+a}\right).$$

5. **A5 (BEL)** Let f be any function that maps the set of real numbers into the set of real numbers. Prove that there exist real numbers x and y such that

$$f(x-f(y))>yf(x)+x.$$

6. **A6 (USA)** IMO3 Suppose that $s_1, s_2, s_3, \dots$ is a strictly increasing sequence of positive integers such that the subsequences

$$s_{s_1}, s_{s_2}, s_{s_3}, \dots \quad \text{and} \quad s_{s_1+1}, s_{s_2+1}, s_{s_3+1}, \dots$$

are both arithmetic progressions. Prove that the sequence $s_1, s_2, s_3, \dots$ is itself an arithmetic progression.

7. **A7 (JAP)** Find all functions f from the set of real numbers into the set of real numbers which satisfy for all real x, y the identity

$$f(xf(x+y)) = f(yf(x)) + x^2.$$

8. **C1 (NZL)** Consider 2009 cards, each having one gold side and one black side, lying in parallel on a long table. Initially all cards show their gold sides. Two players, standing by the same long side of the table, play a game with alternating moves. Each move consists of choosing a block of 50 consecutive cards, the leftmost of which is showing gold, and turning them all over, so those with showed gold now show black and vice versa. The last player who can make a legal move wins.
(a) Does the game necessarily end?
(b) Does there exist a winning strategy for the starting player?

9. **C2 (ROM)** For any integer $n \geq 2$, let $N(n)$ be the maximal number of triples (a_i, b_i, c_i), $i = 1, \dots, N(n)$, consisting of nonnegative integers a_i, b_i, and c_i such that the following two conditions are satisfied:
(i) $a_i + b_i + c_i = n$ for all $i = 1, \dots, N(n)$,
(ii) If $i \neq j$, then $a_i \neq a_j$, $b_i \neq b_j$, and $c_i \neq c_j$.
Determine $N(n)$ for all $n \geq 2$.

10. **C3 (RUS)** Let n be a positive integer. Given a sequence $\varepsilon_1, \dots, \varepsilon_{n-1}$ with $\varepsilon_i = 0$ or $\varepsilon_i = 1$ for each $i = 1, \dots, n-1$, the sequences $a_0, \dots, a_n$ and $b_0, \dots, b_n$ are constructed by the following rules:

$$a_0 = b_0 = 1, \ \ a_1 = b_1 = 7,$$

$$a_{i+1} = \begin{cases} 2a_{i-1} + 3a_i, & \text{if } \varepsilon_i = 0, \\ 3a_{i-1} + a_i, & \text{if } \varepsilon_i = 1, \end{cases} \quad \text{for } i = 1, \dots, n-1,$$

$$b_{i+1} = \begin{cases} 2b_{i-1} + 3b_i, & \text{if } \varepsilon_{n-i} = 0, \\ 3b_{i-1} + b_i, & \text{if } \varepsilon_{n-i} = 1, \end{cases} \quad \text{for } i = 1, \dots, n-1.$$

Prove that $a_n = b_n$.

11. **C4 (NET)** For an integer $m \geq 1$ we consider partitions of a $2^m \times 2^m$ chessboard into rectangles consisting of cells of the chessboard, in which each of the 2^m cells along one diagonal forms a separate rectangle of side length 1. Determine the smallest possible sum of rectangle perimeters in such a partition.

12. **C5 (NET)** Five identical empty buckets of 2-liter capacity stand at the vertices of a regular pentagon. Cinderella and her wicked Stepmother go through a sequence of rounds: At the beginning of every round the Stepmother takes one liter of water from the nearby river and distributes it arbitrarily over the five buckets. Then Cinderella chooses a pair of neighboring buckets, empties them into the river, and puts them back. Then the next round begins. The Stepmother's goal is to make one of these buckets overflow. Cinderella's goal is to prevent this. Can the wicked Stepmother enforce a bucket overflow?

13. **C6 (BUL)** On a 999×999 board a *limp rook* can move in the following way: From any square it can move to any of its adjacent squares, i.e. a square having a common side with it, and every move must be a turn: i.e. the directions of any two consecutive moves must be perpendicular. A *non-intersecting route* of the limp rook consists of a sequence of pairwise different squares that the limp rook can visit in that order by an admissible sequence of moves. Such a non-intersecting route is called *cyclic*, if the limp rook can, after reaching the last square of the route, move directly to the first square of the route and start over. How many squares does the longest possible cyclic, non-intersecting route of a limp rook visit?

14. **C7 (RUS)** IMO6 Let $a_1, a_2, \dots, a_n$ be distinct positive integers and let M be a set of $n-1$ positive integers not containing $s = a_1 + a_2 + \cdots + a_n$. A grasshopper is to jump along the real axis, starting at the point 0 and making n jumps to the right with lengths $a_1, a_2, \dots, a_n$ in some order. Prove that the order can be chosen in such a way that the grasshopper never lands on any point in M.

15. **C8 (AUT)** For any integer $n \geq 2$ we compute the integer $h(n)$ by applying the following procedure to its decimal representation. Denote by r the rightmost digit of n.
   1° If $r = 0$, then the decimal representation of $h(n)$ results from the decimal representation of n by removing this rightmost digit 0.
   2° If $1 \leq r \leq 9$ we split the decimal representation of n into a maximal right part R that solely consists of digits not less than r and into the left part L that either is empty or ends with a digit strictly smaller than r. Then the decimal representation of $h(n)$ consists of the decimal representation of L, followed by two copies of the decimal representation of $R-1$. For instance, for the number $n = 17,151,345,543$ we will have $L = 17,151$, $R = 345,543$, and $h(n) = 17,151,345,542,345,542$.

   Prove that, starting with an arbitrary integer $n \geq 2$, iterated application of h produces the integer 1 after finitely many steps.

16. **G1 (BEL)** IMO4 Let ABC be a triangle with $AB = AC$. The angle bisectors of $\angle CAB$ and $\angle ABC$ meet the sides BC and CA at D and E, respectively. Let K be the incenter of triangle ADC . Suppose that $\angle BEK = 45°$. Find all possible values of $\angle CAB$.

17. **G2 (RUS)** IMO2 Let ABC be a triangle with circumcenter O. The points P and Q are interior points of the sides CA and AB, respectively. Let K, L and M be the

midpoints of the segments BP, CQ, and PQ, respectively, and let Γ be the circle passing through K, L, and M. Suppose that the line PQ is tangent to the circle Γ. Prove that $OP = OQ$.

18. **G3 (IRN)** Let ABC be a triangle. The incircle of ABC touches the sides AB and AC at the points Z and Y, respectively. Let G be the point where the lines BY and CZ meet, and let R and S be the points such that the two quadrilaterals $BCYR$ and $BCSZ$ are parallelograms. Prove that $GR = GS$.

19. **G4 (GBR)** Given a cyclic quadrilateral $ABCD$, let the diagonals AC and BD meet at E and the lines AD and BC meet at F. The midpoints of AB and CD are G and H, respectively. Show that EF is tangent at E to the circle through the points E, G, and H.

20. **G5 (POL)** Let P be a polygon that is convex and symmetric with respect to some point O. Prove that for some parallelogram R satisfying $P \subseteq R$ we have
$$\frac{|S_R|}{|S_P|} \leq \sqrt{2}.$$

21. **G6 (UKR)** Let the sides AD and BC of the quadrilateral $ABCD$ (such that AB is not parallel to CD) intersect at point P. Points O_1 and O_2 are the circumcenters and points H_1 and H_2 are the orthocenters of the triangles ABP and DCP, respectively. Denote the midpoints of segments O_1H_1 and O_2H_2 by E_1 and E_2, respectively. Prove that the perpendicular from E_1 on CD, the perpendicular from E_2 on AB, and the line H_1H_2 are concurrent.

22. **G7 (IRN)** Let ABC be a triangle with incenter I and let X, Y, and Z be the incenters of the triangles BIC, CIA, and AIB respectively. Let the trianlge XYZ be equilateral. Prove that ABC is equilateral too.

23. **G8 (BUL)** Let $ABCD$ be a circumscribed quadrilateral. Let g be a line through A which meets the segment BC in M and the line CD in N. Denote by I_1, I_2, and I_3 the incenters of $\triangle ABM$, $\triangle MNC$, and $\triangle NDA$, respectively. Show that the orthocenter of $\triangle I_1I_2I_3$ lies on g.

24. **N1 (AUS)** IMO1 Let n be a positive integer and let $a_1, \ldots, a_k$ ($k \geq 2$) be distinct integers in the set $\{1, \ldots, n\}$ such that n divides $a_i(a_{i+1} - 1)$ for $i = 1, \ldots, k-1$. Prove that n does not divide $a_k(a_1 - 1)$.
Original formulation: A social club has n members. They have the membership numbers 1, 2, …, n, respectively. From time to time members send presents to other members including items they have already received as presents from other members. In order to avoid the embarrassing situation that a member might receive a present that he or she has sent to other members, the club adds the following rule to its statutes at one of its annual general meetings: "A member with membership number a is permitted to send a present to a member with membership number b if and only if $a(b-1)$ is a multiple of n." Prove that, if each member follows this rule, none will receive a present from another member that he or she has already sent to other members.

Alternative formulation: Let G be a directed graph with n vertices $v_1, v_2, \ldots, v_n$ such that there is an edge oing from v_a to v_b if and only if a and b are distinct and $a(b-1)$ is a multiple of n. Prove that this graph does not contain a directed cycle.

25. **N2 (PER)** A positive integer N is called *balanced*, if $N = 1$ or if N can be written as a product of an even number of not necessarily distinct primes. Given positive integers a and b, consider the polynomial P defined by $P(x) = (x+a)(x+b)$.
    (a) Prove that there exist distinct positive integers a and b such that all the numbers $P(1)$, $P(2)$, $\ldots$, $P(50)$ are balanced.
    (b) Prove that if $P(n)$ is balanced for all positive integers n, then $a = b$.

26. **N3 (EST)** Let f be a non-constant function from the set of positive integers into the set of positive integers, such that $a-b$ divides $f(a) - f(b)$ for all distinct positive integers a and b. Prove that there exist infinitely many primes p such that p divides $f(c)$ for some positive integer c.

27. **N4 (PRK)** Find all positive integers n such that there exists a sequence of positive integers a_1, a_2, $\ldots$, a_n satisfying:
$$a_{k+1} = \frac{a_k^2 + 1}{a_{k-1} + 1} - 1$$
for every k with $2 \le k \le n-1$.

28. **N5 (HUN)** Let $P(x)$ be a non-constant polynomial with integer coefficients. Prove that there is no function T from the set of integers into the set of integers such that the number of integers x with $T^n(x) = x$ is equal to $P(n)$ for every $n \ge 1$, where T^n denotes the n-fold application of T.

29. **N6 (TUR)** Let k be a positive integer. Show that if there exists a sequence a_0, a_1, $\ldots$ of integers satisfying the condition
$$a_n = \frac{a_{n-1} + n^k}{n} \quad \text{for all } n \ge 1,$$
then $k-2$ is divisible by 3.

30. **N7 (MON)** Let a and b be distinct integers greater than 1. Prove that there exists a positive integer n such that $(a^n - 1)(b^n - 1)$ is not a perfect square.

2

Solutions

2.1 Solutions to the Shortlisted Problems of IMO 2009

1. Notice that b_{2009}, r_{2009}, and w_{2009} always form a triangle. In order to prove this fact it suffices to show that the largest of these three numbers (say w_{2009}) is less than the sum of the other two. Since there are b_i and r_j such that w_{2009}, b_i, and r_j form a triangle we get $w_{2009} < b_i + r_j \le b_{2009} + r_{2009}$. This proves that $k \ge 1$.
We will now provide an example in which only one triple (b_i, r_i, w_i) forms a triangle. Let us set $w_i = 2i$, $r_i = i$, for $i = 1, 2, \dots, 2009$; and let us define $b_i = i$ for $i \in \{1, 2, \dots, 2008\}$, and $b_i = 2i$ for $i = 2009$. We will now form triangles $T_1, \dots, T_{2009}$ so that each has one blue, one red, and one white side. For $j = 1, 2, \dots, 2008$ we can define the triangle T_j to be the one with the sides w_j, r_{j+1}, and b_j because $2j < j + j + 1$. The sides of T_{2009} can be $w_{2009} = 4018$, $r_1 = 1$, and $b_{2009} = 4018$. The conditions of the problem are clearly satisfied.

2. The given condition on a, b, and c imply that $abc(a+b+c) = ab+bc+ca$. Applying the inequality $(X+Y+Z)^2 \ge 3(XY+YZ+ZX)$ to $X = ab$, $Y = bc$, $Z = ca$ gives us $(ab+bc+ca)^2 \ge 3abc(a+b+c) = 3(ab+bc+ca)$ which means that $ab+bc+ca \ge 3$.
From $2a+b+c \ge 2\sqrt{(a+b)(a+c)}$ and two analogous inequalities we deduce:
$$\frac{1}{(2a+b+c)^2} + \frac{1}{(2b+c+a)^2} + \frac{1}{(2c+a+b)^2} \le \frac{2(a+b+c)}{4(a+b)(b+c)(c+a)} \le \frac{9(a+b+c)}{16(ab+bc+ca)(a+b+c)}.$$
The last inequality follows from $9(a+b)(b+c)(c+a) = 9(a^2b+ab^2+b^2c+bc^2+c^2a+ca^2)+18abc = 8(ab+bc+ca)(a+b+c)+(a^2b+ab^2+b^2c+bc^2+c^2a+ca^2-6abc) \ge 8(ab+bc+ca)(a+b+c)$.
The required inequality now follows from $ab+bc+ca \ge 3$. The equality holds if and only if $a = b = c = 1$.

3. Assume that f satisfies the given requirements. Let us prove that $f(1) = 1$. Assume that $K = f(1) - 1 > 0$. Let m be the minium of f, and b any number for which $f(b) = m$. Since 1, $m = f(b)$, and $f(b + f(1) - 1) = f(b+K)$ form a triangle we must have $f(b+K) < 1 + f(b)$. The minimality of m implies $f(b+K) = m$, and by induction $f(b+nK) = m$ for all $n \in \mathbb{N}$. There exists a triangle with sides $b+nK$, $f(1)$, and $f(m)$, hence $b+nK < f(1)+f(m)$ for each n. This contradiction implies $f(1) = 1$.
The numbers a, $1 = f(1)$, and $f(f(a))$ form a triangle for every a. Therefore $a-1 < f(f(a)) < a+1$, hence $f(f(a)) = a$ and f is a bijection. We now have that $f(a)$, $f(b)$, and $f(b+a-1)$ determine a triangle for all $a, b \in \mathbb{N}$.
Let $z = f(2)$. Clearly, $z > 1$. Since $f(z)$, $f(z)$, and $f(2z-1)$ form a triangle we get $f(2z-1) < f(z)+f(z) = 2f(f(2)) = 4$. This implies that $f(2z-1) \in \{1,2,3\}$. Since f is a bijection and $f(1) = 1$, $f(z) = 2$, we must have $f(2z-1) = 3$. Let us prove that $f(k) = (k-1)z - k + 2$ for all $k \in \mathbb{N}$.
The statement is true for 1 and 2. Assume that it holds for all of 1, 2, …, k. Since $f((k-1)z-k+2)$, $f(z)$, and $f(kz-k+1)$ form a triangle we have $f(kz-k+$

$1) \le k+1$. The function f is injective hence $f(kz-k+1) \neq i$ unless $kz-k+1 = (i-1)z-i+2$, i.e. $k+1=i$. Therefore $f(kz-k+1)=k+1$, or $f(k+1) = kz-k+1$ and the induction is complete. Furthermore, f is increasing.
If $z>2$ then $2 = f(z) > f(2) = z$, a contradiction. Thus $z=2$ and $f(k) = 2(k-1)-k+2 = k$. It is easy to verify that $f(k)=k$ satisfies the given condition.

4. We first prove that for $a,b>0$ the following inequality holds:

$$\sqrt{\frac{a^2+b^2}{2}}+\sqrt{ab} \le a+b. \tag{1}$$

After dividing both sides by $\sqrt{ab}$ and substituting $\sqrt{\frac{a}{b}} = x$ it becomes

$$\frac{1}{\sqrt{2}}\sqrt{x^2+\frac{1}{x^2}}+1 \le \left(x+\frac{1}{x}\right).$$

Taking squares transforms this inequality into $\left(x+\frac{1}{x}-2\right)^2 \ge 0$, which obviously holds. The equality occurs if and only if $x=1$, or $a=b$.
Rewriting (1) in the form $\sqrt{\frac{a^2+b^2}{a+b}}+\sqrt{2}\sqrt{\frac{ab}{a+b}} \le \sqrt{2}\sqrt{a+b}$ and summing it with the two analogous inequalities for the pairs (b,c) and (c,a) we obtain:

$$\begin{aligned}&\sqrt{\frac{a^2+b^2}{a+b}}+\sqrt{\frac{b^2+c^2}{b+c}}+\sqrt{\frac{c^2+a^2}{c+a}}\\ &+\sqrt{2}\left(\sqrt{\frac{ab}{a+b}}+\sqrt{\frac{bc}{b+c}}+\sqrt{\frac{ca}{c+a}}\right) \le \sqrt{2}\left(\sqrt{a+b}+\sqrt{b+c}+\sqrt{c+a}\right).\end{aligned}$$

It suffices to prove that $\sqrt{\frac{ab}{a+b}}+\sqrt{\frac{bc}{b+c}}+\sqrt{\frac{ca}{c+a}} \ge \frac{3}{\sqrt{2}}$. Applying the mean inequality $M_1 \ge M_{-2}$ to the sequence $\sqrt{\frac{ab}{a+b}}, \sqrt{\frac{bc}{b+c}}, \sqrt{\frac{ca}{c+a}}$ gives us

$$\begin{aligned}\sqrt{\frac{ab}{a+b}}+\sqrt{\frac{bc}{b+c}}+\sqrt{\frac{ca}{c+a}} &\ge 3\sqrt{\frac{3}{\frac{a+b}{ab}+\frac{b+c}{bc}+\frac{c+a}{ca}}}\\ &= 3\cdot\sqrt{\frac{3abc}{2(ab+bc+ca)}} \ge \frac{3}{\sqrt{2}}.\end{aligned}$$

This completes the proof of the required inequality. The equality holds if and only if $a=b=c$.

5. Assume the contrary, that $f(x-f(y)) \le yf(x)+x$ for all $x,y \in \mathbb{R}$. Setting $y=0$, $x = z+f(0)$ gives $f(z) \le z+f(0)$. For $x=f(y)$ we get $f(0) \le yf(f(y))+f(y) \le yf(f(y))+y+f(0)$ hence we can conclude that $y(f(f(y))+1) \ge 0$. This means that if $y>0$ then $f(f(y)) \ge -1$, and if $y<0$ then $f(f(y)) \le -1$.
If $f(x)>0$ for some x, then each $y<x-f(0)$ must satisfy $f(y)-f(0) \le y < x-f(0)$, therefore $f(y)<x$. From here we conclude that $-1 \le f(f(x-f(y))) \le$

$f(x-f(y))+f(0)\leq yf(x)+x+f(0)$ and $y\geq\frac{-1-x-f(0)}{f(x)}$. We concluded that each $y<x-f(0)$ must be greater than or equal than $\frac{-1-x-f(0)}{f(x)}$ which is impossible. Therefore $f(x)\leq 0$ for all x. In particular, for all real x we have $f(x)\leq x+f(0)\leq x$. For any $z>0$ we now have $f(-1)=f[(f(z)-1)-f(z)]\leq zf(f(z)-1)+f(z)-1\leq z(f(z)-1)+f(z)-1=(z+1)(f(z)-1)\leq -z-1$. Thus each $z>0$ satisfies $z\leq -f(-1)-1$ which is a contradiction.

6. Assume that $s_{s_n}=an+b$ and $s_{s_n+1}=cn+d$ for some $a,b,c,d\in\mathbb{Z}$. If $m>n>0$ are two integers then $s_m-s_n=(s_{n+1}-s_n)+(s_{n+2}-s_{n+1})+\cdots+(s_m-s_{m-1})\geq m-n$ because the sequence s_n is increasing. Hence $s_{n+1}-s_n\leq s_{s_{n+1}}-s_{s_n}=a$. Denote by m and M the minimal and maximal value of $s_{n+1}-s_n$ as $n\in\mathbb{N}$. Our goal is to prove that $M=m$. Assume the contrary, $m<M$. If $s_{k+1}-s_k=m$ for some $k\in\mathbb{N}$ we get $a=s_{s_{k+1}}-s_{s_k}=(s_{s_k+1}-s_{s_k})+(s_{s_k+2}-s_{s_k+1})+\cdots+(s_{s_{k+1}}-s_{s_{k+1}-1})\leq m\cdot M$. Similarly if $s_{l+1}-s_l=M$ for some $l\in\mathbb{N}$ we get that $a=s_{s_{l+1}}-s_{s_l}\geq m\cdot M$. In particular these two inequalities imply that:

$$\begin{aligned} M\cdot m&=a,\\ s_{s_k+1}-s_{s_k}&=M,\ \text{ whenever } s_{k+1}-s_k=m, \text{ and}\\ s_{s_l+1}-s_{s_l}&=m,\ \text{ whenver } s_{l+1}-s_l=M. \end{aligned}$$

Take any $k\in\mathbb{N}$ such that $s_{k+1}-s_k=m$. Then $M=s_{s_k+1}-s_{s_k}=ck+d-(ak+b)=(c-a)k+d-b$. Furthermore, we have $m=s_{s_{s_k}+1}-s_{s_{s_k}}=(c-a)s_k+d-b$. Repeating the same argument yields $M=(c-a)s_{s_k}+d-b$. Consequently the equation $(c-a)x+d-b=M$ has two solutions $x=k$ and $x=s_{s_k}$ which yields $s_k=k$. Since $s_k=s_1+(s_2-s_1)+\cdots+(s_k-s_{k-1})\geq k$ we conclude that $s_i=i$ for $i=1,2,\dots,k$. Thus $M=s_{s_k+1}-s_{s_k}=s_{k+1}-s_k=m$, a contradiction.

7. Substituting $x=0$ in the given relation gives $f(0)=f(yf(0))$ for all y. Therefore $f(0)=0$, because otherwise for each $z\in\mathbb{R}$ we could take $y=z/f(0)$ to get $f(z)=f(0)$ meaning that f is constant (that is obviously impossible). We now have $f(xf(x))=f(xf(x+0))=f(0f(x))+x^2=x^2$ and $0=f(xf(x-x))=f(-xf(x))+x^2$ implying $f(-xf(x))=-x^2$. Hence f is onto.
If $f(z)=0$ for some $z\neq 0$ we would have $0=f(zf(z))=z^2$, a contradiction.
Assume that $f(x)=f(y)$ for some $x,y\in\mathbb{R}$. Then $x^2=f(xf(x))=f(xf(y))=f((y-x)f(x))+x^2$ giving that $f((y-x)f(x))=0$ hence $f(x)=0$ or $x-y=0$. Both cases now yield $x=y$, therefore f is one-to-one.
Now we will prove that $f(-x)=-f(x)$ for all $x\in\mathbb{R}$. Assume that $x\neq 0$ (the other case is trivial). If $f(x)>0$ there exists z such that $f(x)=z^2$. Since f is injective and $f(zf(z))=z^2$ we conclude that $x=zf(z)$ hence $f(-x)=f(-zf(z))=-z^2=-f(x)$. The case $f(x)<0$ is similar.
On the other hand we have:

$$\begin{aligned} f(yf(x))&=-x^2+f(xf(x+y))=-x^2+(x+y)^2-[(x+y)^2+f(-xf(x+y))]\\ &=y^2+2xy-f((x+y)f(y))=2xy+[(-y)^2+f((x+y)f(-y))]\\ &=2xy+f(-yf(x)) \end{aligned}$$

which implies that $f(xf(y)) = xy$. Analogously, $f(yf(x)) = xy$ hence $xf(y) = yf(x)$. Hence $f(x) = cx$ for some $c \in \mathbb{R}$. The equation $f(xf(x)) = x^2$ implies that $c \in \{-1, 1\}$. Clearly, both $f(x) = x$ and $f(x) = -x$ satisfy the given conditions.

8. (a) If we denote gold cards by 1, and black by 0, the entire sequence of cards corresponds to a number in binary representation. After each of the moves, the number decreases, hence the game has to end.
   (b) We will show that second player wins a game no matter how the players play. Consider the cards whose position (counted from the right) is divisible by 50. There is a total of 40 such cards, and in each move exactly one of this cards is turned over. In the beginning, all 40 of these cards are 1, and in the end all 40 are 0, hence the second player must win.

9. Assume that $(a_i, b_i, c_i)_{i=1}^{N}$ satisfy the conditions of the problem. Then

$$\sum_{i=1}^{N} a_i \geq \frac{N(N-1)}{2},$$

and similarly the two analogous inequalities hold for the sequences (b_i) and (c_i). Hence $3N(N-1)/2 \leq \sum_{i=1}^{N}(a_i + b_i + c_i) = nN$ which implies that $N \leq \left[\frac{2n}{3}\right] + 1$. To prove that there are sequences of length $\left[\frac{2n}{3}\right] + 1$ with the given properties let us consider the following cases:

1° $n = 3k$ for some $k \in \mathbb{N}$. We can take $(a_i, b_i, c_i) = (i-1, k+i-1, 2k-2i+2)$ for $i = 1, 2, \dots, k+1$, and $(a_i, b_i, c_i) = (3k-i+2, 2k-i+1, 2(i-k)-3)$ for $i = k+2, \dots, 2k+1$.

2° $n = 3k+1$ for some $k \in \mathbb{N}$. Take $(a_i, b_i, c_i) = (i-1, k+i-1, 2k-2i+3)$ for $i = 1, 2, \dots, k+1$, and $(a_i, b_i, c_i) = (3k-i+2, 2k-i+1, 2(i-k)-2)$ for $i = k+2, \dots, 2k+1$.

3° $n = 3k-1$ for some $k \in \mathbb{N}$. Define $(a_i, b_i, c_i) = (i-1, k+i-1, 2k-2i+1)$ for $i = 1, 2, \dots, k$, and $(a_i, b_i, c_i) = (3k-i+1, 2k-i, 2i-2k-2)$ for $i = k+1, \dots, 2k$.

10. For a binary sequence $(\varepsilon)_{n-1} = (\varepsilon_i)_{i=1}^{n-1}$, let us define $f(u, v, (\varepsilon)_{n-1}) = c_n$ where the sequence $(c_i)_{i=1}^{n}$ is defined as: $c_0 = u$, $c_1 = v$, and

$$c_{i+1} = \begin{cases} 2c_{i-1} + 3c_i, & \text{if } \varepsilon_i = 0, \\ 3c_{i-1} + c_i, & \text{if } \varepsilon_i = 1, \end{cases} \quad \text{for } i = 1, \dots, n-1.$$

The given sequences (a_n) and (b_n) can be now rewritten as $a_n = f(1, 7, (\varepsilon)_{n-1})$, $b_n = f(1, 7, (\overline{\varepsilon})_{n-1})$ where $(\overline{\varepsilon})_{i=1}^{n-1}$ is defined as $\overline{\varepsilon}_i = \varepsilon_{n-i}$. Using the induction on n we will prove that $f(1, 7, (\varepsilon)_{n-1}) = f(1, 7, (\overline{\varepsilon})_{n-1})$. This is straight-forward to verify for $n = 2$ and $n = 3$, so assume that $n > 3$ and that the statement is true for all binary sequences $(\varepsilon)_k$ of length smaller than n. Notice that $f(\alpha u_1 + \beta u_2, \alpha v_1 + \beta v_2, (\varepsilon)_m) = \alpha f(u_1, v_1, (\varepsilon)_m) + \beta f(u_2, v_2, (\varepsilon)_m)$ (this easily follows by induction on m). Assuming that $\varepsilon_n = 0$, we obtain:

$$\begin{aligned} f(1,7,(\varepsilon)_n) &= 2f(1,7,(\varepsilon)_{n-2}) + 3f(1,7,(\varepsilon)_{n-1}) \\ &= 2f(1,7,\overline{(\varepsilon)_{n-2}}) + 3f(1,7,\overline{(\varepsilon)_{n-1}}) \\ &= 2f(1,7,\overline{(\varepsilon)_{n-2}}) + 3f(7,f(1,7,(\varepsilon_{n-1})_1),\overline{(\varepsilon)_{n-2}}) \\ &= f(23,14+3f(1,7,(\varepsilon_{n-1})_1),\overline{(\varepsilon)_{n-2}}). \end{aligned}$$

Using $14+3f(1,7,(\varepsilon_{n-1})_1) = f(1,7,(0,\varepsilon_{n-1})_2)$ and $23 = f(1,7,(0)_1)$ we get

$$\begin{aligned} & f(23,14+3f(1,7,(\varepsilon_{n-1})_1),\overline{(\varepsilon)_{n-2}}) \\ = \ & f(f(1,7,(0)_1),f(1,7,(0,\varepsilon_{n-1})_2),\overline{(\varepsilon)_{n-2}}) \\ = \ & f(1,7,\overline{(\varepsilon)}_n). \end{aligned}$$

To finish the proof, it remains to see that for $\varepsilon_n = 1$ we have:

$$\begin{aligned} f(1,7,(\varepsilon)_n) &= 3f(1,7,\overline{(\varepsilon)_{n-2}}) + f(1,7,\overline{(\varepsilon)_{n-1}}) \\ &= f(f(1,7,(1)_1),f(1,7,(1,\varepsilon_{n-1})_2),\overline{(\varepsilon)_{n-2}}) \\ &= f(1,7,(\overline{\varepsilon})_n). \end{aligned}$$

11. Denote by (i,j) the cells of the table, and assume that the diagonal cells (i,i) form separate rectangles. We will prove by induction that it is possible to have a perimeter of $p_m = (m+1)2^{m+2}$. The case $m = 0$ is obvious, and assume that the statement holds for some $m \geq 0$. Divide the $2^{m+1} \times 2^{m+1}$ board into four equal boards. Each of the two off-diagonal squares has perimeter $4 \cdot 2^m$ while the other two can be partitioned into rectangles of total perimeter p_m each. The total perimeter is therefore equal to $2 \cdot 4 \cdot 2^m + 2p_m = (m+2)2^{m+3}$.
Let us now prove the other direction, that the total perimeter P satisfies $P \geq (m+1)2^{m+2}$. Assume that the table is partitioned into n rectangles in the described way. Denote by R_i the set of those rectangles that contain at least one square from the ith row. Similarly, let C_i be the set of rectangles that contain the squares from the ith column. Clearly the intersection $C_i \cap R_i$ contains only the diagonal square (i,i). We certainly have

$$P = 2\left(\sum_{i=1}^{2^m} |R_i| + \sum_{i=1}^{2^m} |C_i|\right).$$

Let $\mathscr{F}$ be the collection of all subsets of rectangles in the partition. Since there are n rectangles, we have $|\mathscr{F}| = 2^n$. Let $\mathscr{F}_i$ denote the collection of those subsets S that satisfy: $(R_i \setminus (i,i)) \subseteq S$, and $C_i \cap S \subseteq \{(i,i)\}$. Since $\mathscr{F}_1$, $\mathscr{F}_2$, $\dots$, $\mathscr{F}_{2^m}$ are pairwise disjoint and $|\mathscr{F}_i| = 2^{n-|C_i|-|R_i|+2}$, using the Jensen's inequality applied to $f(x) = 2^{-x}$ we obtain:

$$2^n \geq 2^{n+2}\sum_{i=1}^{2^m} 2^{-|C_i|-|R_i|} \geq 2^{n+2} \cdot 2^m \cdot 2^{-\frac{1}{2^m}\sum(|C_i|+|R_i|)} = 2^{n+m+2-\frac{1}{2^m}\cdot\frac{P}{2}}.$$

This yields to $\frac{P}{2} \geq (m+1) \cdot 2^{m+1}$ which is the relation we wanted to prove.

12. We will show that Cinderella can always make sure that after each of her moves:
(i) The total amount of water in all buckets is less than $3/2$; and
(ii) The amount of water in each pair of non-adjacent buckets is smaller than 1.
The condition (ii) ensures that the Stepmother won't be able to make a bucket overflow. Both (i) and (ii) hold in the beginning of the game. Assume that they are satisfied after the kth round, and let us prove that in the round $k+1$ Cinderella can make them both hold again. Denote the buckets by 1, ..., 5 (counterclockwise), and let x_i be the amount of water in the bucket i. After the $k+1$st move of the Stepmother we have $x_1+\cdots+x_5<\frac{5}{2}$ and $x_i+x_{i+2}<2$ for each i (summation of indeces is modulo 5). It is impossible that $x_i+x_{i+2}\geq 1$ for each i hence we may assume that $x_2+x_5<1$. If $x_1+x_2+x_5<\frac{3}{2}$, it would be safe for Cinderella to empty the buckets 3 and 4. Assume therefore that $x_1+x_2+x_5\geq\frac{3}{2}$. Hence $x_1>\frac{1}{2}$. If both $x_2+x_4\geq 1$ and $x_3+x_5\geq 1$ hold then we must have $x_1\leq\frac{1}{2}$, a contradiction. Assume therefore that $x_2+x_4<1$. If $x_2+x_3+x_4<\frac{3}{2}$ she could empty 1 and 5, so assume that $x_2+x_3+x_4\geq\frac{3}{2}$. This gives $x_3>\frac{1}{2}$. $x_2+\frac{5}{2}>(x_1+x_2+x_5)+(x_2+x_3+x_4)\geq 6$ which yields $x_2>\frac{1}{2}$. Therefore at least one of $x_1+x_4<1$ or $x_3+x_5<1$ holds, say the first one. Thus, if Cinderella empties the buckets 2 and 3 the condition (ii) will be satisfied. (i) will hold as well because $x_2+x_3>\frac{1}{2}+\frac{1}{2}>1$.

13. Let $k\geq 3$. We will prove that the longest cyclic route in a $(4k-1)\times(4k-1)$ board has length $4\cdot[(2k-1)^2-1]$. Let us label the cells with 1, 2, 3, 4 using the pattern from the figure 1, so that the top-left corner is labeled by 1.

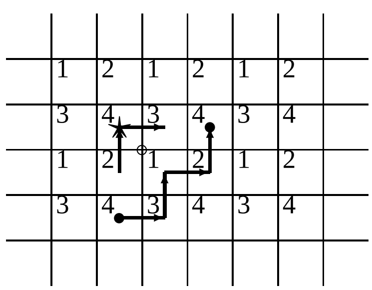


Figure 1

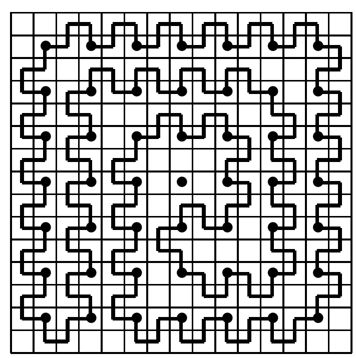


Figure 2

Any four consecutive jumps lend on squares with different labels. Therefore, each cyclic path has equal number of squares of each label. The label 4 appears exactly $(2k-1)^2$ times, but we will prove now that the limp rook can't visit all of them. Assume the contrary – that it is possible for a cyclic route to pass through all the squares labeled by 4. If we paint all such squares alternately black and white so that the top left square is black, we see that the number of black squares is by 1 bigger than the number of white ones. Therefore, a cyclic route has two consecutive black squares. Assume that these squares are those denoted by $\bullet$.

Without loss of generality we may assume that the part of the route is as shown in the figure 1. Since the route always visits $4,3,1,2$ in that order, immediately before visiting $\star$, the rook has to land on the cell labeled by 2 that is exactly below $\star$. The rook has to leave $\star$ by visiting 3 exactly to the right of it. Each point of the two-dimensional plane must be all the time either to the left or to the right of the rook when it is passing next to it. However, this is not the case with the point marked by $\circ$. A contradiction. Therefore the rook can visit at most $4\cdot((2k-1)^2-1)$ squares in a cyclic route. The figure 2 shows how it is possible to recursively make a route that only omits the central square and visits all the other squares labeled by 4.

14. We will prove the statement by induction. The case $n=1$ is trivial, so let us assume that $n>1$ and that the statement holds for 1, 2, ..., $n-1$. Assume that $a_1<\cdots<a_n$. Let $m\in M$ be the smallest element. Consider the following cases:
 1° $m<a_n$: If $a_n\not\in M$ then if the grasshopper makes the first jump of size a_n the problem gets reduced to the sequence a_1, ..., a_{n-1} and the set $M\setminus\{m\}$, which immediately follows by induction. Let us assume that $a_n\in M$. Consider the following $n-1$ pairs: (a_1,a_1+a_n), ..., $(a_{n-1},a_{n-1}+a_n)$. All numbers from these pairs belong to the $n-2$-element set $M\setminus\{a_n\}$, hence one of these pairs, say (a_k,a_k+a_n), has both of its members outside of M. If the first two jumps of the grasshopper are a_k, and a_k+a_n, it has jumped over at least two members of M: m and a_n. There are at most $n-3$ more elements of M to jump over, and $n-2$ more jumps, so the claim follows by induction.
 2° $m\geq a_n$: By induction hypothesis the grasshopper can start from the point $s=a_1+\cdots+a_n$, make $n-1$ jumps of sizes a_1, ..., a_{n-1} to the left, and avoid all the points of $M\setminus\{m\}$. If it misses the point m as well, then we are done (first make a jump of size a_n and reverse the previously made jumps). Suppose that after making the jump a_k the grasshopper landed at site m. If it changes the jump a_k to the jump a_n, it will miss the site m and all subsequent jumps will lend outside of M because m is the left-most point.

15. For each $i=0,1,\dots,9$ denote by N_i the set of all finite strings whose terms are from $\{i,i+1,\dots,9\}$ (the empty string ϕ belongs to each of N_i). Define functions $m_i:N_i\to\mathbb{N}$ recursively: For each $x\in N_9$ we set $m_9(x)=1+$ the number of digits of x (set $m_9(\phi)=1$). Once m_9, ..., m_{i+1} are defined we construct m_i as: Write each $x\in N_i$ in the form $x=\overline{x_0ix_1i\cdots x_{t-1}ix_t}$ where $x_0,\dots,x_t\in N_{i+1}$, and let

$$m_i(x)=\sum_{s=0}^{t}4^{m_{i+1}(x_s)}.$$

Let us prove that $m=m_0$ satisfies $m(h(n))<m(n)$ whenever $n\neq\phi$. If the last digit of n is 0, then $n=\overline{l_00l_1\cdots0l_t0}$ for some $l_0,\dots,l_t\in N_1$ and $m(n)-m(h(n))=4^{m_1(\phi)}>0$. Assume that $n=\overline{Ler(d+1)}$ for $9\geq d\geq e\geq 0$ and $r\in N_{d+1}$. If $L\in N_i$ for some $i<e$ then $L=\overline{l_0il_1i\cdots il_t}$ with $l_0,\dots,l_t\in N_{i+1}$. Denote $n'=\overline{l_ter(d+1)}$. Then $h(n')=\overline{l_terdrd}$ and $m_i(n)-m_i(h(n))=4^{m_{i+1}(n')}-4^{m_{i+1}(h(n'))}$. To prove that $m(n)>m(h(n))$ it suffices to prove that $m_{i+1}(n')>m_{i+1}(h(n'))$.

Repeating this argument we reduce our problem to the case $i = e$. There are l_0, $\dots$, l_t, $r \in N_{d+1}$ such that $n = \overline{l_0 e \cdots e l_t e r(d+1)}$. For $e = d$ we need to prove that $0 < m_d(n) - m_d(h(n)) = 4^{m_{d+1}(\overline{r(d+1)})} - 2 \cdot 4^{m_{d+1}(r)}$. This inequality suffices even in the case $e < d$ because $r \in N_{d+1}$ implies

$$m_{e+1}(\overline{r(d+1)}) = 4^{\cdots 4^{m_d(\overline{r(d+1)})}}, \text{ and } m_{e+1}(\overline{rdrd}) = 4^{\cdots 4^{m_d(\overline{rdrd})}}$$

where 4 appears $d - e - 1$ times in the exponents. If $d + 1 = 9$ then $m_d(n) - m_d(h(n)) = 4^{k+1} - 2 \cdot 4^k > 0$ where k is the number of digits of r. If $d < 8$ then we can write $r = \overline{r_0(d+1)\cdots(d+1)r_s}$ for some $r_0, \cdots, r_s \in N_{d+2}$. We get $m_{d+1}(r) = 4^{m_{d+2}(r_0)} + \cdots + 4^{m_{d+2}(r_s)}$ and $m_{d+1}(\overline{r(d+1)}) = m_{d+1}(r) \cdot 4^{4^{m_{d+2}(\phi)}}$. Hence $m_d(n) - m_d(h(n)) = 4^{m_{d+1}(r)}(4^{4^{m_{d+2}(\phi)}} - 2) > 0$.
Therefore the function m is positive and decreasing on the sequence n, $h(n)$, $h(h(n))$, $\dots$, forcing this sequence to eventually become equal to ϕ. The only way this can occur is if the last terms of the sequence are 1, 00, 0, ϕ.

16. Denote by I and L the incenters of $\triangle ABC$ and $\triangle BDA$. From $\angle ALI = \angle LBA + \angle LAB = 45^\circ$ we see that $AL \| EK$. Let L' be the intersection of DK and BI. From $\angle DL'I = \angle BID - \angle IDK = \angle A/4 = \angle LAI$ we conclude that A, L, D, and L' belong to a circle. Hence $\angle LAL' = 180^\circ - \angle LDL' = 90^\circ$. Now consider $\triangle AKL'$. The segment KE is the altitude from K and $\angle KAE = \angle KL'E$. We will now prove that this is equivalent to E being the orthocenter or $\triangle AKL'$ being isosceles (with $KA = KL'$). Denote by P and Q the intersections of $L'E$ and AE with AK and KL' respectively. If $KA = KL'$ then the statement is obvious. If this is not the case, then PQ intersects AL' at some point M. Then KE is the polar line of the point M with respect to the circumcircle k of $PQL'A$, and since $MA \perp KE$ we conclude that MA contains the center of k. Then we must have $\angle APL' = \angle AQL' = 90^\circ$.
If E is the orthocenter of $\triangle KL'A$ then from $\triangle ABP$ we conclude that $3\angle A/4 + 45^\circ - \angle A/4 = 90^\circ$ which yields to $\angle A = 90^\circ$. If $KA = KL'$, this together with $KA = AL = AL'$ implies that $\triangle AKL'$ is equilateral and $\angle A/4 = \angle KL'E = 60^\circ - \angle LL'A = 15^\circ$. That means that $\angle A = 60^\circ$.
It is easy to verify that $\angle A \in \{60^\circ, 90^\circ\}$ implies $\angle BEK = 45^\circ$.

17. From $MK \| AB$ and $ML \| AC$ we get $\angle KML = \angle BAC$. Also, $\angle AQP = \angle KMQ = \angle MLK$, because of the assumption that PQ is a tangent to Γ. Therefore $\triangle AQP \sim \triangle MLK$ hence

$$\frac{AQ}{AP} = \frac{ML}{MK} = \frac{PC}{QB},$$

and $AQ \cdot QB = AP \cdot PC$. The quantity on the left-hand side of the last equality represents the power of the point Q with respect to the circumcircle of $\triangle ABC$ and it is equal to $OA^2 - OQ^2$. Similarly, $AP \cdot PC = OA^2 - OP^2$. Thus $OA^2 - OP^2 = OA^2 - OQ^2$ implying $OP = OQ$.

18. Consider the excircle k_a corresponding to the vertex A. Let X be the point of tangency of the incircle k and BC, and X_a the point of tangency of k_a and BC. Similarly, let us denote by Z_a and Y_a the points of tangency of k_a with AB and

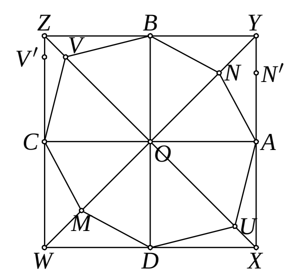
AC. Then we have $ZZ_a = ZB + BZ_a = BX + BX_a = BX + CX = BC = ZS$. We used the fact that $BX_a = CX$. On the other hand, $CY_a = CX_a = BX = BZ = CS$. Denote by s the degenerated circle with center S and radius 0. Points Z and C belong to the radical axis of the circles s and k_a. Similarly, we prove that B and Y belong to the radical axis of the circles k_a and r, where r is the circle with center R and radius 0. Thus G is the radical center of r, s, and k_a, hence $GS = GR$.

19. Denote by M and N respectively the points symmetric to E with respect to G and H. Notice that $\angle FAM = \angle FAC + \angle BEC = \angle EBC + \angle CEB = \angle FCE$. From $\triangle FDC \sim \triangle FBA$ we have $\frac{FA}{FC} = \frac{AB}{CD}$ and from $\triangle AEB \sim \triangle DEC$ we have $\frac{AB}{DC} = \frac{BE}{CE}$. Therefore

$$\frac{FA}{FC} = \frac{BE}{CE} = \frac{AM}{CE}$$

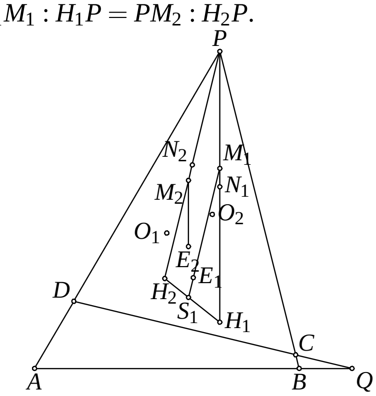
hence $\triangle FAM \sim \triangle FCE$ and $\angle AFM = \angle EFC$. Similarly we prove that $\triangle FDN \sim \triangle FBE$ and $\angle AFN = \angle EFC$. This implies that F, N, and M are colinear. Since H and G are the midpoints of EN and EM it suffices to show that FE is the tangent to the circumcircle of $\triangle NEM$. From $\triangle FDN \sim \triangle FBE$ we have $\frac{FD}{FB} = \frac{FN}{FE}$, while the similarity $\triangle FAM \sim \triangle FCE$ implies that $\frac{FC}{FA} = \frac{FE}{FM}$. Using $\triangle FDC \sim \triangle FBA$ again, we finally obtain $\frac{FD}{FB} = \frac{FC}{FA}$. Thus $\frac{FE}{FM} = \frac{FN}{FE}$ and FE is tangent to the circumcircle of $\triangle MEN$.

20. Let A and B be two vertices of the polygon P for which $S_{\triangle AOB}$ is maximal. Let C and D be the points symmetric to A and B with respect to O. Let $WXYZ$ be the parallelogram such that A, B, C, and D are the midpoints of XY, YZ, ZW, and WX, and $WX \| OA$, $XY \| OB$. The polygon P is contained inside $WXYZ$. Let U, V, M, and N be the intersections of P with XZ and WY such that the order of points on lines is $W-M-N-Y$ and $X-U-V-Z$. There are two parallel lines u and v through U and V such that P is within the strip between u and v; similarly, there are two parallel lines $n \ni N$ and $m \ni M$ such that P is within the strip between these two lines. The lines u, v, n, and m determine another parallelogram $EFGH$. We will prove that $S_{EFGH} \le \sqrt{2} S_P$ or $S_{WXYZ} \le \sqrt{2} S_P$.



By performing affine transformations to the plane, the ratios of the areas of the figures don't change, and we can choose the transformations in such a way that $WXYZ$ maps to a square. Then $EFGH$ maps to a rectangle, and P maps to another convex polygon. We may thus assume that $WXYZ$ was a square to start with, and $EFGH$ was a rectangle. Let V' be the projection of V to WZ, and let N' be the projection of N to YX. Denote $a = OA$, $x = ZV'$, and $y = YN'$. Then $S_P \ge S_{ANBVCMDU} = 4S_{\triangle COV} + 4S_{\triangle OAN} = 2a(a-x) + 2a(a-y) = 2a(2a-(x+y))$. We also have $S_{WXYZ} = 4a^2$ and $S_{EFGH} = 4OV \cdot ON = 8(a-x)(a-y)$. If we assume that $S_{WXYZ} > \sqrt{2} S_P$ and $S_{EFGH} > \sqrt{2} S_P$, multiplying these two inequalities gives us: $32a^2(a-x)(a-y) > 8a^2(2a-(x+y))^2$, which after simplification becomes $4xy > (x+y)^2$, or equivalently $(x-y)^2 < 0$, a contradiction.

21. Assume that the perpendicular from E_1 to CD and the perpendicular from E_2 to AB intersect H_1H_2 at S_1 and S_2 respectively. We will prove that $H_1S_1 : H_1H_2 = H_1S_2 : H_1H_2$ which will imply $S_1 = S_2$. Let $M_1 = S_1E_1 \cap PH_1$ and $M_2 = S_2E_2 \cap PH_2$. It suffices to establish the relation $H_1M_1 : H_1P = PM_2 : H_2P$.



Let us denote by N_1 and N_2 the midpoints of PH_1 and PH_2. Without loss of generality, assume that M_1 is between P and N_1, and consequently N_2 is between P and M_2. Our goal is to prove that $\frac{H_1M_1}{H_1N_1} = \frac{PM_2}{PN_2}$, or after subtracting 1 from both sides:

$$\frac{M_1N_1}{H_1N_1} = \frac{M_2N_2}{PN_2}. \qquad (1)$$

From $E_1M_1 \| PH_2$ and $E_2M_2 \| PH_1$ we conclude that $\angle N_2M_2E_2 = 180^\circ - \angle E_1M_1N_1$. Observe that $E_1N_1 \| PO_1$ hence $\angle M_1E_1N_1 = \angle O_1PH_2 = \angle DPH_2 - \angle APO_1$. From $\triangle DCP$ we obtain the equality $\angle DPH_2 = 90^\circ - \angle CDP$ and from $\triangle ABP$ we have $\angle APO_1 = 90^\circ - \angle ABP$. Therefore $\angle M_1E_1N_1 = \angle ABP - \angle CDP$. In a similar way we prove that $\angle M_2E_2N_2 = \angle O_2PH_1 = \angle ABP - \angle CDP$ which gives us $\angle M_1E_1N_1 = \angle M_2E_2N_2$. Applying the sine theorem to $\triangle E_2N_2M_2$ and $\triangle E_1N_1M_1$ we get $\frac{M_1N_1}{E_1N_1} = \frac{M_2N_2}{E_2N_2}$. Hence in order to prove (1) we need to verify that $\frac{E_1N_1}{H_1N_1} = \frac{E_2N_2}{PN_2}$, or equivalently, $\frac{PO_1}{PH_1} = \frac{PO_2}{PH_2}$. From $PH_1 = 2O_1X$ where X is the midpoint of AB we see that $\frac{PO_1}{PH_1} = \frac{AO_1}{2O_1X} = \frac{1}{2\cos\angle AO_1X} = \frac{1}{2\cos\angle APB}$. Analogously we prove that $\frac{PO_2}{PH_2} = \frac{1}{2\cos\angle CPD}$ and this completes the proof of the required statement.

22. Let us denote by α, β, and γ the angles of $\triangle ABC$. Then we calculate $\angle BIX = \angle XIC = \frac{1}{2}\angle BIC = 45^\circ + \frac{\alpha}{4}$ and similar two formulas hold for $\angle ZIX$ and $\angle ZIY$. If we denote by P and Q the feet of perpendiculars from I to CX and CY then $IP = IQ$. Since $\angle PIX = \angle PIC - \angle XIC = (90^\circ - \angle XCI) - 45^\circ - \frac{\alpha}{4} = 45^\circ - \frac{\alpha}{4} - \frac{\gamma}{4} = \frac{\beta}{4}$, and similarly $\angle QIY = \frac{\alpha}{4}$, we deduce that $IX/\cos\frac{\alpha}{4} = IY/\cos\frac{\beta}{4}$. If we denote the previous quantity by ρ, we get $IX = \rho\cos\frac{\alpha}{4}$, $IY = \rho\cos\frac{\beta}{4}$, and analogously, $IZ = \rho\cos\frac{\gamma}{4}$. Applying the cosine theorem to $\triangle ZIX$ gives us that $ZX^2 - ZI^2 = IX^2 - 2IX\cdot ZI\cdot\cos\left(90^\circ + \frac{\alpha+\gamma}{4}\right) = \rho^2\left(\cos^2\frac{\alpha}{4} + 2\cos\frac{\alpha}{4}\cos\frac{\gamma}{4}\sin\frac{\alpha+\gamma}{4}\right)$. Using the similar relation for $ZY^2 - ZI^2$ and the assumption $ZX = ZY$ we get:

$$\begin{aligned} 0 &= \cos^2\frac{\alpha}{4} - \cos^2\frac{\beta}{4} + 2\cos\frac{\alpha}{4}\sin\frac{\alpha}{4}\cos^2\frac{\gamma}{4} - 2\cos\frac{\beta}{4}\sin\frac{\beta}{4}\cos^2\frac{\gamma}{4} \\ &\quad + 2\cos^2\frac{\alpha}{4}\sin\frac{\gamma}{4}\cos\frac{\gamma}{4} - 2\cos^2\frac{\beta}{4}\sin\frac{\gamma}{4}\cos\frac{\gamma}{4} \\ &= \left(\cos^2\frac{\alpha}{4} - \cos^2\frac{\beta}{4}\right)\cdot\left(1 - \sin\frac{\gamma}{2}\right) + \cos^2\frac{\gamma}{4}\left(\sin\frac{\alpha}{2} - \sin\frac{\beta}{2}\right). \end{aligned}$$

We now use the formulas $\cos^2 X - \cos^2 Y = \frac{1}{2}(\cos(2X) - \cos(2Y)) = -\sin(X + Y)\sin(X - Y)$, and $\sin X - \sin Y = 2\sin\frac{X-Y}{2}\cos\frac{X+Y}{2}$ to obtain:

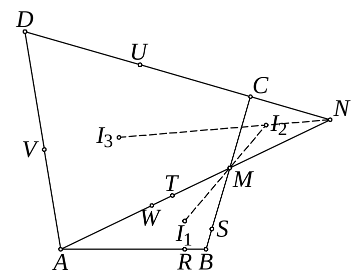
$$0 = \sin\frac{\alpha-\beta}{4}\left(2\cos^2\frac{\gamma}{4}\cos\frac{\alpha+\beta}{4} - \left(1-\sin\frac{\gamma}{2}\right)\sin\frac{\alpha+\beta}{4}\right). \qquad (1)$$

Let E be the expression from the last parenthesis. Since $\frac{\alpha+\beta}{4} = 45^\circ - \frac{\gamma}{4}$, then:

$$\begin{aligned}\sqrt{2}E &= \left(1+\cos\frac{\gamma}{2}\right)\left(\cos\frac{\gamma}{4}+\sin\frac{\gamma}{4}\right) - \left(1-\sin\frac{\gamma}{2}\right)\left(\cos\frac{\gamma}{4}-\sin\frac{\gamma}{4}\right)\\ &= 2\sin\frac{\gamma}{4}+\cos\frac{\gamma}{2}\cos\frac{\gamma}{4}-\sin\frac{\gamma}{2}\sin\frac{\gamma}{4}+\cos\frac{\gamma}{2}\sin\frac{\gamma}{4}+\sin\frac{\gamma}{2}\cos\frac{\gamma}{4}.\end{aligned}$$

Hence $\sqrt{2}E = 2\sin\frac{\gamma}{4}+\cos\frac{3\gamma}{4}+\sin\frac{3\gamma}{4} = 2\sin\frac{\gamma}{4}+\sqrt{2}\sin\left(45^\circ+\frac{3\gamma}{4}\right)$. Clearly, the last quantity is positive as the sin is positive function on $(0, 180^\circ)$. Thus from (1) we get $\alpha = \beta$. Similarly, we prove that $\beta = \gamma$ and $\triangle ABC$ is equilateral.

23. Denote by k_1, k_2, and k_3 the incircles of $\triangle ABM$, $\triangle MNC$, and $\triangle ADN$, respectively. Let R, S, T be the points of tangency of k_1 with AB, BM, and MA; U, V, W the tangency points of k_3 with ND, DA, and AN; and P and Q the points of tangency of tangents from C to k_3 and k_1 different than CD and CB, respectively. Assume that the configuration of the points is as in the picture. From



$CD + AB = CB + DA$ we get $CU + UD + AR + RB = DV + VA + BS + SC$ which together with $DU = DV$, $AR = AW + WT = AV + WT$, and $BR = BS$ implies that $CU + WT = CS = CQ$. On the other hand $CU + WT = CP + WT \geq CP + PQ$ because $PQ \leq WT$ and the equality holds if and only if PQ is a common tangent of the circles k_1 and k_3. We conclude that $CQ \geq CP + PQ$. The triangle inequality yields $CQ = CP + PQ$ hence C, P, and Q are colinear.

We now have that $\angle I_3CI_1 = \frac{1}{2}\angle DCB = 90^\circ - \angle I_2CM = \angle I_3I_2M$, hence C belongs to the circle circumscribed about $\triangle I_1I_2I_3$. The Simson's line corresponding to C bisects the segment CH, where H is the orthocenter of $\triangle I_1I_2I_3$. It remains to notice that the line g is the image of the Simson's line under the homothety with center C and coefficient 2. Indeed, the reflections of C with respect to I_1I_2 and I_2I_3 belong to g because I_1I_2 and I_2I_3 are the bisectors of $\angle CMN$ and $\angle CNM$.

24. Assume the contrary that $a_ia_{i+1} \equiv a_i \pmod{n}$, for $i = 1, 2, \ldots, k$ (summation of indices is modulo k). The case $k = 2$ is trivial as we have $a_1a_2 \equiv a_1$ and $a_1a_2 \equiv a_2$ which yields to immediate contradiction $a_1 \equiv a_2$. Here and in the sequel, all of the congruences are modulo n. Assume that $1 < i < k$. Multiplying the congruence $a_ia_{i+1} \equiv a_i$ by $a_1 \cdots a_{i-1}$ we get $a_1 \cdots a_{i+1} \equiv a_1 \cdots a_i$. By induction we get that $a_1 \cdots a_k \equiv a_1$. Since everything is cyclic in analogous way we obtain $a_1 \cdots a_k \equiv a_2$ which yields to $a_1 \equiv a_2$ and this is a contradiction.

25. (a) To each $n \in \mathbb{N}$ we can correspond a sequence $(s_1, \ldots, s_{50})$ of numbers from $\{0, 1\}$ such that

$$s_i = \begin{cases} 0, \text{ if } n+i \text{ is balanced,} \\ 1, \text{ if } n+i \text{ is not balanced.} \end{cases}$$

Since there are at most 2^{50} sequences that correspond to natural numbers, we see that there are $a, b \in \mathbb{N}$ that correspond to the same sequence. For such a choice of a and b the number $P(i)$ is balanced for each $i \in \{1, 2, \ldots, 50\}$.

(b) Assume that $a < b$ and that $P(n) = (n+a)(n+b)$ is balanced for each $n \in \mathbb{N}$. For each $k > a$, consider the number $x = k(b-a) - a$. Then $P(x) = (b-a)^2 k(k+1)$ is balanced, which means that k and $k+1$ are *equibalanced* (if one is balanced then so is the other) whenever $k > a$. Then all numbers greater than a are equibalanced which can't be true, as squares are balanced but primes are not.

26. Assume the contrary: There are finitely many prime numbers $p_1, p_2, \ldots, p_m$ such that no other prime can be a divisor of $f(n)$ as $n \in \mathbb{N}$. Assume that $\alpha_1, \ldots, \alpha_m$ are non-negative integers for which $f(1) = p_1^{\alpha_1} \cdots p_m^{\alpha_m}$. For any sequence $\beta = (\beta_1, \ldots, \beta_m)$ satisfying $\beta_1 > \alpha_1, \ldots, \beta_m > \alpha_m$, consider the number $a_\beta = p_1^{\beta_1} \cdots p_m^{\beta_m}$. Assume that $f(a_\beta + 1) = p_1^{\gamma_1} \cdots p_m^{\gamma_m}$, for some $\gamma_1, \cdots, \gamma_m \in \mathbb{N}_0$. Since $a_\beta \mid f(a_\beta + 1) - f(1)$, we can conclude that $\gamma_1 = \alpha_1, \ldots, \gamma_m = \alpha_m$ hence $f(a_\beta + 1) = f(1)$. If n is a positive integer for which $f(n) \neq f(1)$, then $a_\beta + 1 - n \mid f(a_\beta + 1) - f(n) = f(1) - f(n)$. This relation has to hold for every sequence β satisfying $\beta_1 > \alpha_1, \ldots, \beta_n > \alpha_1$, which is impossible.

27. We will prove that $n \leq 4$ by showing that there is no sequence of length 5, and that there is a sequence of length 4 satisfying the conditions.
Assume that $a_1, \ldots, a_5$ is a sequence of length 5. If $2 \nmid a_k$ for some $k \leq 3$, from $a_{k+1}^2 + 1 = (a_k + 1)(a_{k+2} + 1)$ we see that $2 \nmid a_{k+1}$ as well. Notice that a_1 and a_2 are even. Indeed, if $2 \nmid a_k$ for $k \in \{1, 2\}$, then $2 \nmid a_{k+1}$ and $2 \nmid a_{k+2}$. We then have $a_{k+1}^2 + 1 \equiv 2 \pmod 4$ while $4 \mid (a_k + 1)(a_{k+2} + 1)$, which is contradiction.
Since a_1 and a_2 are even, we get from $a_2^2 + 1 = (a_1 + 1)(a_3 + 1)$ that a_3 is even as well. We now have $a_3 + 1 \mid a_2^2 + 1$ and $a_2 + 1 \mid a_3^2 + 1$. Let us prove that there are no two positive even integers x and y satisfying $x + 1 \mid y^2 + 1$ and $y + 1 \mid x^2 + 1$. Assume the contrary, that (x, y) is one such pair for which $x + y$ is minimal and $x \geq y$. Let $d = \gcd(x+1, y+1)$. From $d \mid x + 1$ one gets $d \mid x^2 - 1$. Since $d \mid y + 1 \mid x^2 + 1$ we derive that $d \mid (x^2 + 1) - (x^2 - 1) = 2$. Since $x + 1$ is odd we see that $d = 1$. Therefore $x + 1 \mid y^2 + 1 + x^2 - 1 = x^2 + y^2$ and $y + 1 \mid x^2 + y^2$ imply $(x+1)(y+1) \mid x^2 + y^2$. There exists $m \in \mathbb{N}$ such that $m(x+1)(y+1) = x^2 + y^2$. Consider the quadratic polynomial $P(\lambda) = \lambda^2 - m(y+1)\lambda - m(y+1) + y^2$. Since $P(x) = 0$, there exists a positive integer x' such that $P(\lambda) = (\lambda - x)(\lambda - x')$. From $x + x' = m(y+1)$ and $xx' = y^2 - m(y+1)$ we get that x' is even and $y^2 + 1 = (x+1)(x'+1)$. We now must have $x' < y \leq x$, hence (x', y) is another pair of even natural numbers such that $x' + 1 \mid y^2 + 1$ and $y + 1 \mid x'^2 + 1$, a contradiction.
One sequence of length 4 is $a_1 = 4$, $a_2 = 33$, $a_3 = 217$, $a_4 = 1384$.

28. Assume that there exists $T:\mathbb{Z}\to\mathbb{Z}$ and a polynomial P with integer coefficients such that $T^n(x)=x$ has exactly $P(n)$ solutions for each $n\in\mathbb{N}$. For $k\in\mathbb{N}$ denote by $B(k)$ the set of those x such that $T^k(x)=x$ but $T^l(x)\neq x$ for all $0\le l<k$. Take any $x\in A(n)\cap B(k)$, and assume that $n=ak+b$, for $a\in\mathbb{N}_0$ and $0\le b\le k-1$. Then $x=T^n(x)=T^b(T^{ak}(x))=T^b(x)$. We conclude that $b=0$ and $k\mid n$. Hence $A(n)=\bigcup_{k|n}B(k)$ and moreover

$$|A(n)|=\sum_{k|n}|B(k)|.$$

Assume now that $x\in B(n)$, and consider the sequence $\{T^i(x)\}_{i=0}^{n-1}$. Each $T^i(x)$ belongs to $A(n)$ since $T^i(x)=T^i(T^n(x))=T^n(T^i(x))$. If $T^i(x)=T^{i+j}(x)$ for $0\le i\le n-1$ and $0\le j\le n-1$, then $x=T^n(x)=T^{n-i}(T^i(x))=T^{n-i}(T^{i+j}(x))=T^{n+j}(x)=T^j(T^n(x))=T^j(x)$ which means that $j=0$. Therefore, $T^{i_1}(x)\neq T^{i_2}(x)$ whenever $i_1\neq i_2$ and $i_1,\ i_2\in\{0,1,\dots n-1\}$. In addition, each of $T^i(x)$ belongs to $B(n)$. This means that $B(n)$ partitions into sequences of n elements in each, and thus $n\mid B(n)$.
Let p be a prime number. We have $P(p)=|A(p)|=|B(1)|+|B(p)|$. If q is also prime, then $P(pq)=|B(1)|+|B(p)|+|B(q)|+|B(pq)|$ hence $P(pq)\equiv|B(1)|+|B(q)| \pmod p$. However, from $P(pq)\equiv P(0)\pmod p$ we get that $P(0)-|B(1)|-|B(q)|$ is divisible by p. If we fix q, this remains to hold for each prime p. Therefore $P(0)=|B(1)|+|B(q)|=P(q)$. However, this is now true for every prime q, hence P must be constant, contrary to our assumptions.

29. For each $k\in\mathbb{N}$ there exist $q_k\in\mathbb{Z}$ and a polynomial $P_k(x)$ of degree $k-1$ with integer coefficients such that $xP_k(x)=x^k+P_k(x-1)+q_k$. Indeed, the coefficients of $P_k(x)=c_{k-1}x^{k-1}+\cdots+c_0$ form a system of linear equations which we can explicitly solve (we can see that $c_{k-1}=1$). The sequence $b_n=a_n-P_k(n)$ satisfies the recursive relation $b_n=\frac{b_{n-1}}{n}-\frac{q_k}{n}$. Inductively we prove that $b_n=\frac{b_0}{n!}-\frac{q_k}{n!}\cdot\sum_{i=0}^{n-1}i!$, hence $a_n-P_k(n)=\frac{a_0-P_k(0)}{n!}-\frac{q_k}{n!}\cdot\sum_{i=0}^{n-1}i!$. All of $a_n-P_k(n)\in\mathbb{Z}$, and $|a_n-P_k(n)|\le\frac{|a_0-P_k(0)|}{n!}+\frac{|q_k|}{n}+|q_k|\cdot\sum_{i=0}^{\infty}\frac{1}{i^2}\to 0$. Hence we conclude that $a_0=P_k(0)$ and $q_k=0$.
We will finish the proof by showing that q_k is even only when $k\equiv 2\pmod 3$. We start with the equality $\Gamma_k(x)=xP_k(x)-x^k-P_k(x-1)-q_k=0$ and use the fact that $x(x+1)\Gamma_k(x)-\Gamma_{k+1}(x)-\Gamma_{k+2}(x)=0$. After simplifying this becomes equivalent to $xT_k(x)=T_k(x-1)+2xP_k(x-1)+2xq_k-(q_k+q_{k+1}+q_{k+2})$, for $T_k(x)=x(x+1)P_k(x)-P_{k+1}(x)-P_{k+2}(x)-q_kx$. Therefore $xT_k(x)-T_k(x-1)\equiv q_k+q_{k+1}+q_{k+2}\pmod 2$. For each polynomial f one of the two identities hold: either $f(x)\equiv x\pmod 2$ for all $x\in\mathbb{Z}$, or $f(x)\equiv 0\pmod 2$ for all $x\in\mathbb{Z}$. Since $xT_k(x)-T_k(x-1)$ is a constant polynomial modulo 2, it must be 0, and $q_k+q_{k+1}+q_{k+2}=0$. We can easily calculate $q_1=-1$, and $q_2=0$, and now by induction it is straight-forward to establish $q_k\equiv 0\pmod 2$ if and only if $k\equiv 2\pmod 3$.

30. We will prove a stronger statement by assuming that a and b are perfect squares. Assume that each of $\rho_n=\sqrt{(a^n-1)(b^n-1)}$ is an integer. Consider the Taylor

representation of $f(x) = (1-x)^{1/2} = \sum_{k=0}^{\infty} \alpha_k x^k$ for $x \in (-1,1)$ (the sequence α_k is fixed here). There exist real numbers $(c_{k,l})_{k,l\geq 0}$ such that

$$g(x,y) = (1-x)^{\frac{1}{2}}(1-y)^{\frac{1}{2}} = \sum_{k,l\geq 0} c_{k,l} x^k y^l, \quad \text{for all } x,y \in (-1,1). \tag{1}$$

Therefore $\rho_n = \sum_{k,l\geq 0} c_{k,l} \left(\sqrt{ab}/(a^k b^l)\right)^n$. Take $k_0, l_0 \in \mathbb{N}$ for which $a^{k_0} > \sqrt{ab}$ and $b^{l_0} > \sqrt{ab}$. Consider the polynomial $P(x) = \prod_{k=0}^{k_0} \prod_{l=0}^{l_0} (a^k b^l x - \sqrt{ab})$. There are $d_0, \ldots, d_{k_0 l_0} \in \mathbb{Z}$ such that $P(x) = \sum_{i=0}^{k_0 l_0} d_i x^i$. For each $n \geq 0$ denote

$$\sigma_n = \sum_{i=0}^{k_0 l_0} d_i \rho_{n+i} = \sum_{k,l\geq 0} \left(\frac{\sqrt{ab}}{a^k b^l}\right)^n c_{k,l} P\left(\frac{\sqrt{ab}}{a^k b^l}\right) = \sum_{k>k_0 \text{ or } l>l_0} \gamma_{k,l} \left(\frac{\sqrt{ab}}{a^k b^l}\right)^n,$$

where $\gamma_{k,l} = c_{k,l} P\left(\sqrt{ab}/(a^k b^l)\right)$. The series for σ_n is absolutely convergent because it is a finite linear combination of absolutely convergent series in ρ_{n+i}. Since $\sqrt{ab}/(a^k b^l) < \max\{1/a, 1/b\} \leq 1/2$ then $\sigma_n < \frac{1}{2}\sigma_{n-1}$. This means that $\lim_{n\to\infty} \sigma_n = 0$, and since all of σ_n are integers there exists N such that $\sigma_n = 0$ for $n \geq N$. For $n \geq N$ we have $\sum_{i=0}^{k_0 l_0} d_i \rho_{n+i} = 0$.

Assume first that $a^k \neq b^l$ for each pair (k,l) of positive integers. Solving the system of recursive equations for $(\rho_n)_{n\geq N}$ we find constants $e_{k,l}$ for $0 \leq k \leq k_0$, $0 \leq l \leq l_0$ such that $\rho_n = \sum_{k=0}^{k_0} \sum_{l=0}^{l_0} e_{k,l} \left(\sqrt{ab}/(a^k b^l)\right)^n$. This together with (1) implies that if $(x,y) = (1/a^n, 1/b^n)$ for some $n \geq N$ then $\sum_{k=0}^{\infty} \sum_{l=0}^{\infty} c_{k,l} x^k y^l = \sum_{k=0}^{k_0} \sum_{l=0}^{l_0} e_{k,l} x^k y^l$. These two are Taylor series for $g(x,y)$ and have to be the same. Hence $c_{k,l} = e_{k,l}$ if $k \leq k_0$ or $l \leq l_0$. If either $k > k_0$ or $l > l_0$ then $c_{k,l} = 0$. We conclude that $g(x,y)$ has a finite Taylor expansion around 0, which is impossible.

In the case that there are k and l such that $a^k = b^l$, there would exist an integer p such that $a = p^l$, and $b = p^k$. In a similar way we get a contradiction by proving the finiteness of the Taylor expansion of $(1-x^l)^{1/2}(1-x^k)^{1/2}$.

A

Notation and Abbreviations

A.1 Notation

We assume familiarity with standard elementary notation of set theory, algebra, logic, geometry (including vectors), analysis, number theory (including divisibility and congruences), and combinatorics. We use this notation liberally.
We assume familiarity with the basic elements of the game of chess (the movement of pieces and the coloring of the board).
The following is notation that deserves additional clarification.

- $\mathscr{B}(A,B,C)$, $A-B-C$: indicates the relation of *betweenness*, i.e., that B is between A and C (this automatically means that A,B,C are different collinear points).
- $A = l_1 \cap l_2$: indicates that A is the intersection point of the lines l_1 and l_2.
- AB: line through A and B, segment AB, length of segment AB (depending on context).
- $[AB$: ray starting in A and containing B.
- $(AB$: ray starting in A and containing B, but without the point A.
- (AB): open interval AB, set of points between A and B.
- $[AB]$: closed interval AB, segment AB, $(AB) \cup \{A,B\}$.
- $(AB]$: semiopen interval AB, closed at B and open at A, $(AB) \cup \{B\}$.
  The same bracket notation is applied to real numbers, e.g., $[a,b) = \{x \mid a \leq x < b\}$.
- ABC: plane determined by points A,B,C, triangle ABC ($\triangle ABC$) (depending on context).
- $[AB,C$: half-plane consisting of line AB and all points in the plane on the same side of AB as C.
- $(AB,C$: $[AB,C$ without the line AB.

- $\langle \overrightarrow{a}, \overrightarrow{b} \rangle$, $\overrightarrow{a} \cdot \overrightarrow{b}$: scalar product of $\overrightarrow{a}$ and $\overrightarrow{b}$.
- $a, b, c, \alpha, \beta, \gamma$: the respective sides and angles of triangle ABC (unless otherwise indicated).
- $k(O, r)$: circle k with center O and radius r.
- $d(A, p)$: distance from point A to line p.
- $S_{A_1A_2\ldots A_n}$, $[A_1A_2\ldots A_n]$: area of n-gon $A_1A_2\ldots A_n$ (special case for $n = 3$, S_{ABC}: area of $\triangle ABC$).
- $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$: the sets of natural, integer, rational, real, complex numbers (respectively).
- $\mathbb{Z}_n$: the ring of residues modulo n, $n \in \mathbb{N}$.
- $\mathbb{Z}_p$: the field of residues modulo p, p being prime.
- $\mathbb{Z}[x]$, $\mathbb{R}[x]$: the rings of polynomials in x with integer and real coefficients respectively.
- R^*: the set of nonzero elements of a ring R.
- $R[\alpha]$, $R(\alpha)$, where α is a root of a quadratic polynomial in $R[x]$: $\{a + b\alpha \mid a, b \in R\}$.
- X_0: $X \cup \{0\}$ for X such that $0 \notin X$.
- X^+, X^-, $aX + b$, $aX + bY$: $\{x \mid x \in X, x > 0\}$, $\{x \mid x \in X, x < 0\}$, $\{ax + b \mid x \in X\}$, $\{ax + by \mid x \in X, y \in Y\}$ (respectively) for $X, Y \subseteq \mathbb{R}$, $a, b \in \mathbb{R}$.
- $[x]$, $\lfloor x \rfloor$: the greatest integer smaller than or equal to x.
- $\lceil x \rceil$: the smallest integer greater than or equal to x.

The following is notation simultaneously used in different concepts (depending on context).

- $|AB|$, $|x|$, $|S|$: the distance between two points AB, the absolute value of the number x, the number of elements of the set S (respectively).
- (x, y), (m, n), (a, b): (ordered) pair x and y, the greatest common divisor of integers m and n, the open interval between real numbers a and b (respectively).

A.2 Abbreviations

We tried to avoid using nonstandard notation and abbreviations as much as possible. However, one nonstandard abbreviation stood out as particularly convenient:

- w.l.o.g.: without loss of generality.

Other abbreviations include:

- RHS: right-hand side (of a given equation).

- LHS: left-hand side (of a given equation).
- QM, AM, GM, HM: the quadratic mean, the arithmetic mean, the geometric mean, the harmonic mean (respectively).
- gcd, lcm: greatest common divisor, least common multiple (respectively).
- i.e.: in other words.
- e.g.: for example.

B

Codes of the Countries of Origin

ARG	Argentina	HKG	Hong Kong	POL	Poland
ARM	Armenia	HUN	Hungary	POR	Portugal
AUS	Australia	ICE	Iceland	PRK	Korea, North
AUT	Austria	INA	Indonesia	PUR	Puerto Rico
BEL	Belgium	IND	India	ROM	Romania
BLR	Belarus	IRE	Ireland	RUS	Russia
BRA	Brazil	IRN	Iran	SAF	South Africa
BUL	Bulgaria	ISR	Israel	SER	Serbia
CAN	Canada	ITA	Italy	SIN	Singapore
CHN	China	JAP	Japan	SLO	Slovenia
COL	Colombia	KAZ	Kazakhstan	SMN	Serbia and Montenegro
CRO	Croatia	KOR	Korea, South	SPA	Spain
CUB	Cuba	KUW	Kuwait	SVK	Slovakia
CYP	Cyprus	LAT	Latvia	SWE	Sweden
CZE	Czech Republic	LIT	Lithuania	THA	Thailand
CZS	Czechoslovakia	LUX	Luxembourg	TUN	Tunisia
EST	Estonia	MCD	Macedonia	TUR	Turkey
FIN	Finland	MEX	Mexico	TWN	Taiwan
FRA	France	MON	Mongolia	UKR	Ukraine
FRG	Germany, FR	MOR	Morocco	USA	United States
GBR	United Kingdom	NET	Netherlands	USS	Soviet Union
GDR	Germany, DR	NOR	Norway	UZB	Uzbekistan
GEO	Georgia	NZL	New Zealand	VIE	Vietnam
GER	Germany	PER	Peru	YUG	Yugoslavia
GRE	Greece	PHI	Philippines		