1. (10 points) Assume that $X$ and $Y$ are two random variables. What type of object is $X \cdot Y$?
   
   (A) probability measure    (B) event    (C) random variable    (D) number    (E) improperly defined object

2. (10 points) It is known that $S$ and $T$ are independent events that satisfy $P(S) = \frac{12}{23}$ and $P(S \cap T^C) = \frac{5}{23}$. What is $P(T)$?
   
   (A) $\frac{5}{12}$    (B) $\frac{7}{18}$    (C) $\frac{5}{16}$    (D) $\frac{11}{18}$    (E) $\frac{7}{12}$

3. (10 points) $Z$ is a normal random variable with expectation 81 and variance 16. What is $P(Z \geq 5)$?
   
   (A) $\Phi(21)$    (B) $e^{-\frac{1}{2} \cdot 21^2}$    (C) $e^{-\frac{1}{2} \cdot 19^2}$    (D) $\Phi(19)$    (E) $\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} t^2}$

4. (10 points) The moment generating function of the random variable $X$ is given by
   
   $$M_X(t) = \frac{3e^{-3t} + 2 + 2e^{3t} + 9e^{6t}}{16}.$$ 

   If $m$ and $n$ are relatively prime positive integers for which $E[X] = \frac{m}{n}$, what is $m+n$?
   
   Answer: 

5. (10 points) Assume that $E[X] = 3$ and $\text{var}(X) = 37$. Evaluate $E[(X + 9)(X + 2)]$.

   Answer: 

6. (10 points) The covariance matrix for random variables $X$ and $Y$ is given by

$$
\begin{bmatrix}
\text{Cov}(X, X) & \text{Cov}(X, Y) \\
\text{Cov}(Y, X) & \text{Cov}(Y, Y)
\end{bmatrix} =
\begin{bmatrix}
10 & -1 \\
-1 & 9
\end{bmatrix}.
$$

Evaluate $\text{Cov}(9X + 7Y, 5Y + 6)$.

(A) 25 (B) 32 (C) 98 (D) 168 (E) 270

7. (15 points) Assume that the joint probability density function of the random variables $X$ and $Y$ satisfies

$$
f(x, y) = \begin{cases} 
\frac{1}{20}, & \text{if } 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 8 \\
\frac{1}{80}, & \text{if } 1 < x \leq 13 \text{ and } 0 \leq y \leq 4 \\
0, & \text{otherwise.}
\end{cases}
$$

Calculate $P(X + Y \leq 9)$.

(A) $\frac{12}{25}$ (B) $\frac{7}{10}$ (C) $\frac{3}{4}$ (D) $\frac{13}{16}$ (E) $\frac{17}{20}$

8. (15 points) The random variables $X$ and $Y$ have joint bivariate normal distribution. Their expectations satisfy $E[X] = 5$ and $E[Y] = 2$. Their variances satisfy $\text{var}(X) = 60$ and $\text{var}(Y) = 81$. The covariance between $X$ and $Y$ is 63. Calculate $E[X|Y = 29]$.

Answer: 

9. (25 points) A total of $N$ balls is placed into $N$ boxes in such a way that each ball is equally likely to be placed in each of the boxes and the placements are independent from each other. Find the expected number and the variance of the number of empty boxes.
1. The correct answer is C.

2. Since the events $S$ and $T$ are independent we have $P(S \cap T) = P(S) \cdot P(T)$. The events $S \cap T$ and $S \cap T^C$ are disjoint and their union is $S$. Therefore

$$P(S) = P(S \cap T) + P(S \cap T^C).$$

Therefore

$$P(T) = \frac{P(S \cap T)}{P(S)} = \frac{P(S) - P(S \cap T^C)}{P(S)} = \frac{7}{12}.$$

The correct answer is E.

3. There exists a standard normal random variable $W$ such that $Z = 81 + 4W$. The required probability is

$$P(Z \geq 5) = P(81 + 4W \geq 5) = P(W \geq \frac{5 - 81}{4}) = P(-W \leq \frac{5 - 81}{4}) = P(W \leq 19).$$

Since $-W$ is a standard normal random variable as well, the last quantity is equal to $\Phi(19)$. The correct answer is D.

4. The expectation of the random variable can be recovered from the first derivative of the moment generating function as $E[X] = \frac{d}{dt} M_X(0)$. The first derivative of the moment generating function is

$$\frac{d}{dt} M_X(t) = -3 \cdot 3e^{-3t} + 2 \cdot 6e^{6t} + 9 \cdot 9e^{9t}.$$

Placing $t = 0$ gives us

$$\frac{d}{dt} M_X(0) = \frac{-3 \cdot 3 + 2 \cdot 6 + 9 \cdot 9}{16} = \frac{84}{16} = \frac{41}{4}.$$

Thus $m = 41$, $n = 4$ and the answer is 45.

5. The expectation of the required quantity satisfies


6. We will use the bi-linearity of covariance to simplify the required quantity

$$\text{Cov}(9X + 7Y, 5Y + 6) = \text{Cov}(9X + 7Y, 5Y) + \text{Cov}(9X + 7Y, 6).$$

Since 6 is a constant, it is independent of the random variable $9X + 7$ which means that the second covariance is 0. Furthermore,

$$\text{Cov}(9X + 7Y, 5Y + 6) = 9 \cdot 5 \cdot \text{Cov}(X,Y) + 7 \cdot 5 \cdot \text{Cov}(Y,Y) = 5 \cdot (9 \cdot (-1) + 7 \cdot 9) = 270.$$

7. The required probability can be expressed as

$$P(X + Y \leq 9) = P(X \leq 1, Y \leq 8) + P(1 < X \leq 9, Y \leq \min\{9 - X, 4\}).$$

Let us denote by $P_1 = P(X \leq 1, Y \leq 8)$ and $P_2 = P(1 < X \leq 9, Y \leq \min\{9 - X, 4\})$. The first number $P_1$ is calculated as the area of the rectangle with sides 1 and 8 multiplied by the constant $\frac{1}{2}$. The second number $P_2$ is calculated as the product of the constant $\frac{1}{20}$ and the area of the right-angled trapezoid. The trapezoid has the height 4 and its two parallel sides have lengths 8 and 4. Therefore

$$P(X + Y \leq 9) = P_1 + P_2 = \frac{28}{40}.$$

Therefore $P(X + Y \leq 9) = \frac{7}{10}$ and the correct answer is B.
8. There exist independent standard normal random variables $Z$ and $W$ and scalars $\alpha$ and $\beta$ such that $Y = 2 + 9Z$ and $X = 5 + \alpha Z + \beta W$. From $\text{cov}(X, Y) = 63$ we obtain that 

$$63 = \mathbb{E}[9Z \cdot (\alpha Z + \beta W)] = 9 \cdot \alpha + 9 \cdot \beta \cdot \mathbb{E}[ZW] = 9 \alpha.$$ 

Therefore $\alpha = 7$. From $\text{var}(X) = 60$ we get 

$$60 = \mathbb{E}[(\alpha Z + \beta W)^2] = \alpha^2 + \beta^2,$$

we get $\beta^2 = 60 - 49 = 11$. We may take the positive value for $\beta$ which is $\beta = \sqrt{11}$. The required conditional expectation satisfies

$$\mathbb{E}[X|Y = 29] = \mathbb{E}[5 + 7Z + \sqrt{11}W|2 + 9Z = 29] = 5 + 7 \cdot \mathbb{E}\left[Z\left|Z = \frac{29 - 2}{9}\right.\right] + \sqrt{11} \cdot \mathbb{E}\left[W\left|Z = \frac{29 - 2}{9}\right.\right]$$

$$= 5 + 7 \cdot 3 = 26.$$

9. We will introduce the random variables $X_1, \ldots, X_N$ that have values in the set $\{0, 1\}$. The random variable $X_i$ has value 1 if the box $i$ is empty, and value 0 otherwise. The number of empty boxes $N_0$ is a random variable that satisfies

$$N = X_1 + X_2 + \cdots + X_N.$$

The expected value satisfies

$$\mathbb{E}[N_0] = \sum_{i=1}^{N} \mathbb{E}[X_i] = N \mathbb{E}[X_1].$$

Let us evaluate $\mathbb{E}[X_1]$. The probability that each ball misses the box 1 is $\frac{N-1}{N}$, therefore because of independence we have

$$\mathbb{E}[X_1] = \mathbb{P}(X_1 = 1) = \left(\frac{N-1}{N}\right)^N.$$

Thus we obtain

$$\mathbb{E}[N_0] = \frac{(N-1)^N}{N^{N-1}}.$$

The variance satisfies

$$\text{var}(N_0) = \mathbb{E}[N_0^2] - \mathbb{E}[N_0]^2.$$

$$= \sum_{i=1}^{N} \mathbb{E}[X_i^2] + 2 \sum_{i \neq j} \mathbb{E}[X_iX_j] - \frac{(N-1)^{2N}}{N^{2N-2}}.$$

There are $\binom{N}{2}$ terms in the second summation and they are all equal to $\mathbb{E}[X_1X_2]$. Hence

$$\text{var}(N_0) = N \mathbb{E}[X_1^2] + 2 \cdot \binom{N}{2} \mathbb{E}[X_1X_2] - \frac{(N-1)^{2N}}{N^{2N-2}}.$$

Since $X_1 \in \{0, 1\}$ we have $X_1^2 = X_1$. On the other hand $X_1X_2$ has value 1 if and only if both $X_1$ and $X_2$ are 1. Otherwise we must have $X_1X_2 = 0$. The event $\{X_1X_2 = 1\}$ occurs when both boxes 1 and 2 are empty. The probability that each ball misses both $X_1$ and $X_2$ is $\frac{N-2}{N}$. Hence

$$\mathbb{E}[X_1X_2] = \left(\frac{N-2}{N}\right)^N,$$

and we finally obtain that the variance of $N_0$ satisfies

$$\text{var}(N_0) = \frac{(N-1)^N}{N^{N-1}} + \frac{(N-1)(N-2)^N}{N^{N-1}} - \frac{(N-1)^{2N}}{N^{2N-2}}.$$