Bulgarian Mathematical Olympiad 1992, III Round

First Day

1. Prove that the numbers

$$10101, \qquad 1010101, \qquad 10101\cdots101, \qquad \ldots$$

are composite ones.

2. Let a and b positive numbers. Prove thet the inequality $\sqrt{a}+1>\sqrt{b}$ holds if and only if for every $x>1$ the inequality

$$ax+\frac{x}{x-1}>b$$

holds.

3. The hexagon $ABCDEF$ is inscribed in a circle so that $|AB|=|CD|=|EF|$. Let P, Q, R be points of intersection of the diagonals AC and BD, CE and DF, EA and FB respectively. Prove thet the triangles PQR and BDF are similar,

Second day

4. Prove that the sum of squares 3,4,5 or 6 consecutive integers is not a perfect square. Give an example of 11 consecutive integers such that the sum of their squares is a perfect square.

5. A convex 15-gon is given. Prove that at least two of its diagonals lie on lines forming an angle not greater than 2°.

6. Let us denote by $S(x)$ the sum of the digits of the positive integer x in a decimal positional system.

   (a) Prove that for every positive integer x the following inequality hold,

   $$\frac{S(x)}{S(2x)}\leq 5$$

   Is this estimation accurate ?

   (b) Prove thet the function $\frac{S(x)}{S(3x)}$ is unbounded.

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